Distant perturbations of the Laplacian in a multi-dimensional space

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Abstract

We consider the Laplacian in $\mathbb{R}^n$ perturbed by a finite number of distant perturbations those are abstract localized operators. We study the asymptotic behaviour of the discrete spectrum as the distances between perturbations tend to infinity. The main results are the convergence theorem and the asymptotics expansions for the eigenelements. Some examples of the possible distant perturbations are given; they are potential, second order differential operator, magnetic Schrödinger operator, integral operator, and δ-potential.

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Introduction

Spectra of self-adjoint operators with distant perturbations exhibit various interesting features and such operators were studied quite intensively. Much attention was paid to a multiple well Shrödinger operator in the case the wells were separated by a large distance (see, for instance, [6, 11, 8, 9], [4, Sec. 8.6]). The similar problem for the Dirac operator was treated in [7]. The main result of the cited works was the description of the asymptotics behaviour of the isolated eigenvalues as the distances between wells tend to infinity. Recently new problems with more complicated distant perturbations have been considered. S. Kondej and I. Veselić studied a δ-potential supported by a curve which consists of a several components [12]. In the case these components are separated by a large distance their results imply an asymptotic estimate for the lowest spectral gap. The problems with distant perturbations were considered also for the waveguides. In [2] the Dirichlet Laplacian in a planar strip was studied, and the distant perturbations were two segments of the same length on the boundary on which the boundary

condition switched to the Neumann one. The asymptotics expansions for the isolated eigenvalues were constructed as the distance between Neumann segments increased unboundedly. These results were generalized in [3], where we studied Dirichlet Laplacian in a domain formed by two adjacent strips of arbitrary width coupled by two windows. These windows were segments cut out from the common boundary of these strips. The technique employed in [3] followed the general ideas of the paper [1]. In this paper we considered Dirichlet Laplacian in an infinite multi-dimensional tube perturbed by two distant perturbations. The perturbations were two abstract localized operators. The asymptotics expansions for the eigenvalues and the associated eigenfunctions were constructed.

In the present paper we consider the Laplacian in $\mathbb{R}^n$, $n \geqslant 1$, perturbed by several distant perturbations. The number of the perturbations is finite but arbitrary and each perturbation is an abstract localized operator. The restrictions for these operators are quite weak and the results of this paper are applicable to a wide class of distant perturbations of various nature (see Sec. 7).

In the paper we construct the asymptotics expansions for the isolated eigenvalues and the associated eigenfunctions of the problem considered. The technique we develop is a generalization of the approach employed in [1]. Such generalization is needed since the tube considered in [1] was infinite in one dimension only that is not the case for a multi-dimensional space. The main additional ingredient we involve is the technique borrowed from [13, Ch. XIV, Sec. 4]. Our approach allows us actually to reduce the original perturbed operator to a *small regular* perturbation of the direct sum of the limiting operators those are Laplacian with one of the original perturbations. Due to this fact we believe that this approach can be employed not only for the asymptotical purposes, but also in studying other properties of the problems with distant perturbations.

The structure of the paper is as follows. In the next section we formulate the problem and present the main results. In the second section we employ the technique from [13, Ch. XIV, Sec. 4] and transform the equation for the resolvent of the both limiting and perturbed operators to a certain operator equation. We employ it in the third section to obtain an equation for the eigenelements of the perturbed operator. We solve this equation explicitly using the slight modification of the Birman-Schwinger approach suggested in [5]. This allows us to prove the main results in the sixth section. The seventh section is devoted to some examples of the distant perturbation to which the general results of this paper can be applied.

1 Problem and main results

Let $x = (x_1, \ldots, x_n)$ be the Cartesian coordinates in $\mathbb{R}^n$, $n \geqslant 1$. Given any bounded domain $Q \subset \mathbb{R}^n$ by $L_2(\mathbb{R}^n; Q)$ we denote the subset of the functions from $L_2(\mathbb{R}^n)$ whose support lies inside $\overline{Q}$.

Let $\Omega_i \subset \mathbb{R}^n$, $i = 1, \ldots, m$, be bounded non-empty domains with infinitely differentiable boundary. By $\mathcal{L}_i : W_2^2(\Omega_i) \to L_2(\mathbb{R}^n, \Omega_i)$, $i = 1, \ldots, m$, we denote

linear bounded operators satisfying the relations

$$(\mathcal{L}_i u_1, u_2)_{L_2(\Omega_i)} = (u_1, \mathcal{L}_i u_2)_{L_2(\Omega_i)}, \tag{1.1}$$

$$\big|(\mathcal{L}_i u, u)\big| \leqslant c_0 \|\nabla u\|^2_{L_2(\Omega_i)} + c_1 \|u\|^2_{L_2(\Omega_i)} \tag{1.2}$$

for all $u, u_1, u_2 \in W^2_{2,0}(\Omega_i)$, where c_0, c_1 are some constants independent of u, u_1, u_2, and

$$c_0 < 1. \tag{1.3}$$

Since each $u \in W^2_2(\mathbb{R}^n)$ belongs to $W^2_2(\mathbb{R}^n)$, we can regard $W^2_2(\mathbb{R}^n)$ as a subset of $W^2_2(\Omega_i)$. Due to such embedding we can define the operators $\mathcal{L}_i$ on the space $W^2_2(\mathbb{R}^n)$ and consider the operators $\mathcal{L}_i$ as unbounded ones in $L_2(\mathbb{R}^n)$.

We introduce the shift operator in $L_2(\mathbb{R}^n)$ as $\mathcal{S}(a)u := u(\cdot + a)$, where $a \in \mathbb{R}^n$. Let X_i, $i = 1, \ldots, m$, be some points in $\mathbb{R}^n$ and denote $X := (X_1, \ldots, X_m)$, $l_{i,j} := |X_i - X_j|$. We set

$$\mathcal{L}_X := \sum_{i=1}^{m} \mathcal{S}(-X_i)\mathcal{L}_i\mathcal{S}(X_i).$$

This operator is defined on $W^2_2(\Omega_X)$, $\Omega_X := \bigcup\limits_{i=1}^{m} \big(\Omega_i + \{X_i\}\big)$, $\Omega_i + \{X_i\} := \{x : x - X_i \in \Omega_i\}$, and maps this space into $L_2(\mathbb{R}^n; \Omega_X)$. In what follows we assume that the distances between X_i increases unboundedly, i.e., $l_{i,j} \to +\infty$, $i \neq j$. Hence the distances between the domains $\Omega_i + \{X_i\}$ tend to infinity and the operator $\mathcal{L}_X$ can be naturally treated as the distant perturbation formed by the operators $\mathcal{L}_i$, $i = 1, \ldots, m$. We can also consider $\mathcal{L}_X$ as an unbounded one in $L_2(\mathbb{R}^n)$ having $W^2_2(\mathbb{R}^n)$ as the domain.

The main object of our study is the operator $\mathcal{H}_X := -\Delta_{\mathbb{R}^n} + \mathcal{L}_X$ in $L_2(\mathbb{R}^n)$ with the domain $W^2_2(\mathbb{R}^n)$. Here $\Delta_{\mathbb{R}^n}$ denotes the Laplacian in $L_2(\mathbb{R}^n)$ with the domain $W^2_2(\mathbb{R}^n)$. Our main aim is to study the behaviour of the spectrum of $\mathcal{H}_X$ as $l_{i,j} \to +\infty$.

Let $\mathcal{H}_i := -\Delta_{\mathbb{R}^n} + \mathcal{L}_i$ be the operators in $L_2(\mathbb{R}^n)$ having $W^2_2(\mathbb{R}^n)$ as the domain. Throughout the paper we assume that $\mathcal{H}_i$ and $\mathcal{H}_X$ are self-adjoint. By $\sigma(\cdot)$, $\sigma_{\mathrm{ess}}(\cdot)$, $\sigma_{\mathrm{disc}}(\cdot)$ we denote the spectrum, the essential and the discrete spectrum of an operator.

Our first result is as follows.

Theorem 1.1. *The essential spectra of $\mathcal{H}_i$, $\mathcal{H}_X$ coincide with the semi-axis $[0, +\infty)$. The discrete spectra of these operator consist of finitely many negative eigenvalues. The total multiplicity of the isolated eigenvalues of $\mathcal{H}_X$ is bounded uniformly on $l_{i,j}$ provided these lengths are large enough.*

We denote $\sigma_* := \bigcup\limits_{i=1}^{m} \sigma_{\mathrm{disc}}(\mathcal{H}_i)$. We say that $\lambda_* \in \sigma_*$ is $(p_1 + \ldots + p_m)$-multiple if it is a p_i-multiple eigenvalue of $\mathcal{H}_i$, $i = 1, \ldots, m$. The relation $p_i = 0$ corresponds to the case that λ_* is not in the spectrum of $\mathcal{H}_i$. Let $l_X := \min\limits_{i,j} l_{i,j}$.

Theorem 1.2. *Each isolated eigenvalue of* $\mathcal{H}_X$ *converges to zero or to* $\lambda_* \in \sigma_*$ *as* $l_X \to +\infty$. *If* $\lambda_* \in \sigma_*$ *is* $(p_1 + \ldots p_m)$*-multiple, the total multiplicity of the eigenvalues of* $\mathcal{H}_X$ *converging to* λ_* *equals* $p_1 + \ldots p_m$.

Theorem 1.3. *Let* $\lambda_* \in \sigma_*$ *be* $(p_1+\ldots+p_m)$*-multiple, and let* $\lambda_i = \lambda_i(X) \xrightarrow[l_X\to+\infty]{} \lambda_*$, $i = 1, \ldots, p$, $p := p_1 + \ldots + p_m$, *be the eigenvalues of the operator* $\mathcal{H}_X$ *taken counting multiplicity and ordered as follows:*

$$0 \leqslant |\lambda_1(X) - \lambda_*| \leqslant |\lambda_2(X) - \lambda_*| \leqslant \ldots \leqslant |\lambda_p(X) - \lambda_*|.$$

These eigenvalues solve the equation (4.16) and satisfy the asymptotic formulas:

$$\lambda_i(X) = \lambda_* + \tau_i(X)\left(1 + \mathcal{O}\left(l_X^{-\frac{n-3}{2p}} \mathrm{e}^{-l_X\frac{\sqrt{-\lambda_*}}{p}}\right)\right), \quad l \to +\infty. \tag{1.4}$$

Here

$$\tau_i = \tau_i(X) = \mathcal{O}\left(l_X^{-\frac{n-1}{2}} \mathrm{e}^{-l_X\sqrt{-\lambda_*}}\right), \quad l_X \to +\infty,$$

are the zeroes of the polynomial $\det\big(\tau\mathrm{E} - \mathrm{A}(\lambda_*, X)\big)$ *taken counting multiplicity and ordered as follows:*

$$0 \leqslant |\tau_1(X)| \leqslant |\tau_2(X)| \leqslant \ldots \leqslant |\tau_p(X)|.$$

The matrix A *is defined by (4.15). The eigenfunctions associated with* λ_i *obey the asymptotic representation*

$$\psi_i = \sum_{j=1}^{m} \mathcal{S}(-X_j) \sum_{q=1}^{p_j} \kappa^{(i)}_{\alpha_j+q}\psi_{j,q} + \mathcal{O}\big(l_X^{-\frac{n-1}{2}} \mathrm{e}^{-l_X\sqrt{-\lambda_j}}\big), \quad l_X \to +\infty,$$
$$\alpha_1 := 0, \quad \alpha_j := p_1 + \ldots + p_{j-1},$$

in $W_2^2(\mathbb{R}^n)$*-norm. Here* $\psi_{i,j}$, $j = 1, \ldots, p_i$, *are the eigenfunctions of* $\mathcal{H}_i$ *associated with* λ_* *and are orthonormalized in* $L_2(\mathbb{R}^n)$. *The numbers* $\kappa^{(i)}_j$ *are the components of the vectors*

$$\boldsymbol{\kappa}_i = \boldsymbol{\kappa}_i(X) = \begin{pmatrix} \kappa^{(i)}_1(X) \\ \vdots \\ \kappa^{(i)}_p(X) \end{pmatrix},$$

which are the solutions to the system (4.14) for $\lambda = \lambda_i(X)$ *and satisfy the condition*

$$(\boldsymbol{\kappa}_i, \boldsymbol{\kappa}_j)_{\mathbb{C}^p} = \begin{cases} 1, & i = j, \\ \mathcal{O}\left(l_X^{\frac{n-1}{2}} \mathrm{e}^{-l_X\sqrt{-\lambda_*}}\right), & i \neq j. \end{cases} \tag{1.5}$$

As it is stated in this theorem, the leading terms of the asymptotics expansions for the eigenvalues λ_i are determined by the matrix $\mathrm{A}(\lambda_*, X)$. At the same time it could be a difficult problem to calculate this matrix and its eigenvalues explicitly.

In the following theorems we show how to calculate the asymptotics expansions for λ_i in more explicit form.

We will say that a square matrix $\mathrm{A}(X)$ satisfies the condition (A) if it is diagonalizable and the determinant of the matrix formed by the normalized eigenvectors of $\mathrm{A}(X)$ is separated from zero uniformly in $l_{i,j}$ large enough.

Theorem 1.4. *Let the hypothesis of Theorem 1.3 hold true and suppose that the matrix* $\mathrm{A}(\lambda_*, X)$ *can be represented as*

$$\mathrm{A}(\lambda_*, X) = \mathrm{A}_0(X) + \mathrm{A}_1(X), \tag{1.6}$$

where A_0 *satisfies the condition (A) and* $\|\mathrm{A}_1(X)\| \to 0$ *as* $l_X \to +\infty$. *Then the eigenvalues* λ_i *of* $\mathcal{H}_X$ *obey the asymptotic formulas*

$$\lambda_i = \lambda_* + \tau_i^{(0)} \left(1 + \mathcal{O}\left(l_X^{-\frac{n-3}{2}} \mathrm{e}^{-l_X\sqrt{-\lambda_*}}\right)\right) + \mathcal{O}(\|\mathrm{A}_1(l)\|), \quad l_X \to +\infty.$$

Here $\tau_i^{(0)} = \tau_i^{(0)}(X)$ *are the roots of the polynomial* $\det\big(\tau\mathrm{E} - \mathrm{A}_0(X)\big)$ *taken counting multiplicity and ordered as follows:*

$$0 \leqslant |\tau_1^{(0)}(X)| \leqslant |\tau_2^{(0)}(X)| \leqslant \ldots \leqslant |\tau_p^{(0)}(X)|. \tag{1.7}$$

Each of these roots satisfies the estimate

$$\tau_i^{(0)}(X) = \mathcal{O}(\|\mathrm{A}_0(X)\|), \quad l_X \to +\infty.$$

We denote $X_{i,j} := X_i - X_j$.

Theorem 1.5. *Let the hypothesis of Theorem 1.3 holds true. Then the eigenvalues* λ_i *satisfy the asymptotic formulas*

$$\lambda_i(X) = \lambda_* + \tau_i^{(0)}(X) + \mathcal{O}\left(l_X^{-n+2} \mathrm{e}^{-2l_X\sqrt{-\lambda_*}}\right), \quad l_X \to +\infty. \tag{1.8}$$

Here $\tau_i^{(0)}$ *are the roots of the polynomial* $\det\big(\tau\mathrm{E} - \mathrm{A}_0\big)$ *taken counting multiplicity and ordered in accordance with (1.7), and the hermitian matrix* A_0 *reads as follows:*

$$A_{i,j}^{(0)}(X) := \big(\mathcal{L}_k \mathcal{S}(X_{k,r})\psi_{r,s}, \psi_{k,q}\big)_{L_2(\Omega_k)}, \quad \textit{if} \quad k \neq r, \quad A_{i,j}^{(0)}(X) := 0, \quad \textit{if} \quad k = r,$$

where $k = 1, \ldots, m$, $q = 1, \ldots, p_k$, $i = \alpha_k + q$, $r = 1, \ldots, m$, $s = 1, \ldots, p_r$, $i = \alpha_r + s$. *The estimates*

$$\tau_i^{(0)} = \mathcal{O}\left(l_X^{-\frac{n-1}{2}} \mathrm{e}^{-l_X\sqrt{-\lambda_*}}\right), \quad l_X \to +\infty,$$

are valid.

Corollary 1.6. *Let $\lambda_* \in \sigma_*$ be $(1+1+\ldots+0)$-multiple, and ψ_i, $i=1,2$ be the associated eigenfunctions of $\mathcal{H}_i$ normalized in $L_2(\mathbb{R}^n)$. Then the asymptotics expansions for the eigenvalues λ_i, $i=1,2$, are as follows*

$$\lambda_1 = \lambda_* - \left|\left(\mathcal{L}_1\mathcal{S}(X_{1,2})\psi_2, \psi_1\right)_{L_2(\Omega_1)}\right| + \mathcal{O}\left(l_X^{-n+2}\mathrm{e}^{-2l_X\sqrt{-\lambda_*}}\right), \quad l_X \to +\infty,$$
$$\lambda_2 = \lambda_* + \left|\left(\mathcal{L}_1\mathcal{S}(X_{1,2})\psi_2, \psi_1\right)_{L_2(\Omega_1)}\right| + \mathcal{O}\left(l_X^{-n+2}\mathrm{e}^{-2l_X\sqrt{-\lambda_*}}\right), \quad l_X \to +\infty.$$

Theorem 1.7. *Let $\lambda_* \in \sigma_*$ be $(1+0+\ldots+0)$-multiple, and ψ_1 be the associated eigenfunction of $\mathcal{H}_1$ normalized in $L_2(\mathbb{R}^n)$. Then the asymptotic expansion for the eigenvalue $\lambda(X) \xrightarrow[l_X\to+\infty]{} \lambda_*$ of $\mathcal{H}_X$ reads as follows*

$$\lambda(X) = \lambda_* - \sum_{j=2}^{m}\left(\mathcal{L}_1\mathcal{S}(X_{1,j})(\mathcal{H}_j-\lambda_*)^{-1}\mathcal{L}_j\mathcal{S}(X_{j,1})\psi_1, \psi_1\right)_{L_2(\Omega_1)} + \mathcal{O}\big(l_X^{-\frac{3n-5}{2}}\mathrm{e}^{-3l_X\sqrt{-\lambda_*}}\big)$$

as $l_X \to +\infty$. The associated eigenfunction ψ satisfy the asymptotic representation

$$\psi(x,X) = \psi_1(x-X_1) + \mathcal{O}\big(l_X^{-\frac{n-1}{2}}\mathrm{e}^{-l_X\sqrt{-\lambda_*}}\big), \quad l_X \to +\infty.$$

Remark 1.1. In this theorem the operators $(\mathcal{H}_j - \lambda_*)$, $j=2,\ldots,m$, are boundedly invertible since $\lambda_* \notin \sigma_{\mathrm{disc}}(\mathcal{H}_j)$.

In accordance with Theorem 1.4 the leading terms of the asymptotics expansions of the eigenvalues of $\mathcal{H}_X$ can be expressed in terms of the matrix A_0 once it is possible to approximate $\mathrm{A}(\lambda_*, X)$ in the sense of (1.6). One of the possible ways to employ Theorem 1.4 is given in Theorem 1.5. Here the matrix A_0 is calculated explicitly in terms of the limiting eigenfunctions and the operators $\mathcal{L}_i$. We also observe that this matrix is in fact the first-order term in the asymptotic expansion for $\mathrm{A}(\lambda_*, X)$.

One of the general cases is that the number $\lambda_* \in \sigma_*$ is a simple isolated eigenvalue of two of operators $\mathcal{H}_i$. This case is addressed in Corollary 1.6. We stress that in this case the asymptotics expansions for the eigenvalues are very similar to ones for a double-well Schrödinger operator with symmetric wells (see, for instance, [9, Th. 2.8]). At the same time, in our case the number of distant perturbations is arbitrary and no symmetry is assumed.

One more general case is that λ_* is a simple isolated eigenvalue of one of the operators $\mathcal{H}_i$ only. The results for this case are due to Theorem 1.7. In this case Theorem 1.5 does not provide good asymptotics expansions for the eigenvalues of $\mathcal{H}_X$ since the matrix A_0 in this theorem is zero. In view of this fact we have to use second-order term of the asymptotic expansion for $\mathrm{A}(\lambda_*, X)$. We also note that in this case the leading terms in the asymptotics expansion for the eigenvalues of $\mathcal{H}_X$ are smaller by order than leading terms in (1.8).

Generally speaking, some of the eigenvalues of the matrix A_0 in Theorem 1.5 can be identically zero for large $l_{i,j}$. In this case the leading terms in (1.8) vanish. If it occurs, one should employ next-to-leading terms of the asymptotics expansion

for $\mathrm{A}(\lambda_*, X)$ and to treat them as a part of A_0 in (1.6). Such an expansion for $\mathrm{A}(\lambda_*, X)$ can be obtained by the technique employed in the proof of Theorem 1.7. We do not provide such results in the paper in order not to overload the text by quite technical and bulky calculations.

2 Proof of Theorem 1.1

Let $\Omega \subset \mathbb{R}^n$ be a bounded non-empty domain and $\mathcal{L} : W_2^2(\Omega) \to L_2(\mathbb{R}^n; \Omega)$ be an operator satisfying the relations

$$(\mathcal{L}u_1, u_2)_{L_2(\Omega)} = (u_1, \mathcal{L}u_2)_{L_2(\Omega)}, \quad \big|(\mathcal{L}u, u)\big| \leqslant c_0 \|\nabla u\|^2_{L_2(\Omega_\pm)} + c_1 \|u\|^2_{L_2(\Omega)} \tag{2.1}$$

for all $u, u_1, u_2 \in W_2^2(\Omega)$, where c_0, c_1 are some constants, and c_0 obeys (1.3). We introduce the operator $\mathcal{H}_\mathcal{L} := -\Delta_{\mathbb{R}^n} + \mathcal{L}$ in $L_2(\mathbb{R})$ with the domain $W_2^2(\mathbb{R})$ and assume that it is self-adjoint.

Lemma 2.1. $\sigma_{\mathrm{ess}}(\mathcal{H}_\mathcal{L}) = [0, +\infty)$.

Proof. We will employ Weyl criterion to prove the lemma. Let $\lambda \in [0, +\infty)$. By $\chi = \chi(t)$ we denote an infinitely differentiable function cut-off function being one as $r < 0$ and vanishing as $r > 1$. We introduce the sequence of the functions $u_p(x) := c_p |x|^{-n/2+1} J_{n/2-1}(\sqrt{\lambda}|x|)\chi(|x| - p) \in W_2^2(\mathbb{R}^n)$, where J_q is the Bessel function of q-th order. The coefficients c_p are specified by the normalization condition $\|u_p\|_{L_2(\mathbb{R}^n)} = 1$. Since

$$|x|^{-n+2} J^2_{n/2-1}(\sqrt{\lambda}|x|) = \frac{2|x|^{-n+1}}{\pi\sqrt{\lambda}} \left(\cos^2 \left(\sqrt{\lambda}|x| - \frac{(n-3)\pi}{4} \right) + \mathcal{O}(|x|^{-1}) \right),$$

as $|x| \to +\infty$, it follows that $c_p \xrightarrow[p\to+\infty]{} 0$. Using this fact it is easy to check that $\|\mathcal{L}u_p\|_{L_2(\mathbb{R}^n)} \to 0$, $\|\mathcal{H}_\mathcal{L} u_p\|_{L_2(\mathbb{R}^n)} \to 0$ as $p \to +\infty$. Therefore, u_p is a singular sequence for $\mathcal{H}_\mathcal{L}$ at λ and $[0, +\infty) \subseteq \sigma_{\mathrm{ess}}(\mathcal{H}_\mathcal{L})$. The opposite inclusion can be shown completely by analogy with how the same was established in the proof of Lemma 2.1 in [1]. ☐

Lemma 2.2. *The discrete spectrum of the operator $\mathcal{H}_\mathcal{L}$ consists of finitely many negative eigenvalues.*

The proof of this lemma is the same as that of Lemma 2.2 in [1].

We apply now Lemmas 2.1, 2.2 with $\mathcal{L} = \mathcal{L}_i$, $\Omega = \Omega_i$, $i = 1, \ldots, m$ and arrive at the statement of the theorem on $\mathcal{H}_i$. It also follows from Lemmas 2.1, 2.2 with $\mathcal{L} = \mathcal{L}_X$, $\Omega := \Omega_X$, that the essential spectrum of $\mathcal{H}_X$ coincides with $[0, +\infty)$ and the discrete spectrum consists of finitely many eigenvalues. It remains to check that the total multiplicity of these eigenvalues is independent on $l_{i,j}$ provided these lengths are large enough. Completely in the same way how the estimate (2.5) was established in the proof of Lemma 2.2 in [1], one can check that

$$\mathcal{H}_X \geqslant \mathcal{H}_X^{(0)} \oplus \mathcal{H}_X^{(1)}, \tag{2.2}$$

where $\mathcal{H}_X^{(1)}$ is the negative Neumann Laplacian in $\mathbb{R}^n \setminus \Omega_X$, while $\mathcal{H}_X^{(0)}$ denotes the operator

$$-\operatorname{div}\left(1 - c_0 \sum_{i=1}^{m} \chi(|x - X_i| - \varepsilon)\right)\nabla - c_1 \sum_{i=1}^{m} \chi(|x - X_i| - \varepsilon)$$

in Ω_X subject to Neumann boundary condition. Here ε is such that $\Omega_i \subseteq \{x : |x| < \varepsilon\}$, and the lengths $l_{i,j}$ are supposed to be large enough so that supports of $\chi(|x - X_i| - \varepsilon)$ do not intersect for different i. It is clear that $\mathcal{H}_X^{(0)}$ is unitary equivalent to the sum $\bigoplus\limits_{i=1}^{m} \mathcal{H}_{X_i}^{(0)}$, where $\mathcal{H}_{X_i}^{(0)}$ is the operator

$$-\operatorname{div}\big(1 - c_0\chi(|x - X_i| - \varepsilon)\big)\nabla - c_1\chi(|x - X_i| - \varepsilon)$$

in $\{x : |x| < \varepsilon\}$ subject to Neumann boundary condition. This sum is independent on $l_{i,j}$ and has a finite number of negative isolated eigenvalues. By the minimax principle and (2.2) these eigenvalues give the lower bounds for the negative eigenvalues of $\mathcal{H}_X$ that implies that total multiplicity of the negative eigenvalues of $\mathcal{H}_X$ is bounded uniformly on $l_{i,j}$ provided these quantities are large enough.

3 Reduction to an operator equation

In this section we collect some preliminaries which will be employed in the proof of Theorems 1.2-1.7.

Let $\mathcal{L}$ and $\mathcal{H}_{\mathcal{L}}$ be the operators introduced in the previous section. For any $\varepsilon > 0$ by $\mathbb{S}_\varepsilon$ we indicate the set of complex numbers separated from the half-line $[0, +\infty)$ by a distance greater than ε. We also assume that ε is chosen so that $\sigma_{\mathrm{disc}}(\mathcal{H}) \subset \mathbb{S}_\varepsilon$.

Consider the equation

$$(\mathcal{H}_{\mathcal{L}} - \lambda)u = f, \tag{3.1}$$

where $f \in L_2(\mathbb{R}^n; \Omega^\beta)$, $\Omega^\beta := \{x \in \mathbb{R}^n : \operatorname{dist}(\Omega, x) < \beta\}$, $\beta > 0$, $\lambda \in \mathbb{S}_\varepsilon$. We are going to reduce this equation to an operator equation in $L_2(\mathbb{R}^n; \Omega^\beta)$. In order to do it, we will employ the general scheme borrowed from [13, Ch. XIV, Sec. 4].

Let $g \in L_2(\mathbb{R}^n; \Omega^\beta)$ be an arbitrary function. We introduce $v := (-\Delta_{\mathbb{R}^n} - \lambda)^{-1} g$. The function v can be represented as

$$v(x, \lambda) := \int\limits_{\Omega^\beta} G_n(|x - y|, \lambda) g(y)\, \mathrm{d}y, \tag{3.2}$$

$$G_n(t, \lambda) := -\frac{\mathrm{i}^{\frac{n}{2}} (\sqrt[4]{-\lambda})^{n-2}}{2^{\frac{n}{2}+1} \pi^{\frac{n}{2}-1}} t^{-\frac{n}{2}+1} H^{(1)}_{\frac{n}{2}-1}(\mathrm{i}t\sqrt{-\lambda}),$$

where $H^{(1)}_{n/2-1}$ is the Hankel function of the first kind and $(n/2-1)$-th order. The branches of the roots are specified by the requirements $\operatorname{Re}\sqrt{-\lambda} > 0$, $\operatorname{Re}\sqrt[4]{-\lambda} > 0$, $\operatorname{Im}\sqrt[4]{-\lambda} > 0$ as $\lambda \in \mathbb{S}_\varepsilon$.

We denote by $\mathcal{H}_\Omega$ the operator $-\Delta+\mathcal{L}$ in $L_2(\Omega^\beta)$ with domain $W_{2,0}^2(\Omega^\beta)$. Here $W_{2,0}^2(\Omega^\beta)$ consists of the functions from $W_2^2(\Omega^\beta)$ vanishing on $\partial\Omega^\beta$. The operator $\mathcal{H}_\Omega$ is symmetric (see (2.1)), and the operator $(\mathcal{H}_\Omega-\mathrm{i})^{-1}$ is therefore well-defined and is bounded as an operator in $L_2(\Omega^\beta)$. Moreover, $\mathcal{H}_\Omega$ is bounded as an operator from $W_{2,0}^2(\Omega^\beta)$ into $L_2(\Omega^\beta)$. By Banach theorem on inverse operator two last facts imply that the operator $(\mathcal{H}_\Omega-\mathrm{i})^{-1}:L_2(\Omega^\beta)\to W_{2,0}^2(\Omega^\beta)$ is bounded. Using this operator, we define one more function $w:=-(\mathcal{H}_\Omega-\mathrm{i})^{-1}\mathcal{L}v$.

By $\chi_\Omega=\chi_\Omega(x)$ we indicate infinitely differentiable cut-off function being one in $\Omega^{\beta/2}$ and vanishing outside Ω^β. We construct the solution to the equation (3.1) as

$$u(x,\lambda)=\mathcal{T}_1(\lambda)g:=v(x,\lambda)+\chi_\Omega(x)w(x,\lambda). \tag{3.3}$$

This function is obviously an element of $W_2^2(\mathbb{R}^n)$. Now we apply the operator $(\mathcal{H}_\mathcal{L}-\lambda)$ to this function:

$$\begin{aligned}&(\mathcal{H}_\mathcal{L}-\lambda)u=g+\mathcal{L}v+(-\Delta-\lambda+\mathcal{L})\chi_\Omega w=g+\mathcal{T}_2(\lambda)g, \\ &\mathcal{T}_2(\lambda)g:=-2\nabla\chi_\Omega\cdot\nabla w-w(\Delta+\lambda-\mathrm{i})\chi_\Omega.\end{aligned} \tag{3.4}$$

Here we have also used the identities $\mathcal{L}\chi_\Omega w=\mathcal{L}w=\chi_\Omega\mathcal{L}w$. Thus, the equation (3.1) holds true if

$$g+\mathcal{T}_2(\lambda)g=f. \tag{3.5}$$

Lemma 3.1. *The operator $\mathcal{T}_1(\lambda):L_2(\Omega^\beta)\to W_2^2(\mathbb{R}^n)$ is bounded and holomorphic w.r.t. $\lambda\in\mathbb{S}_\varepsilon$. The operator $\mathcal{T}_2(\lambda)$ is bounded in $L_2(\Omega^\beta)$ and holomorphic w.r.t. $\lambda\in\mathbb{S}_\varepsilon$. For each solution of (3.5) the function u defined by (3.3) solves (3.1). And vice versa, for each solution u of (3.1) there exists unique solution g of (3.5) satisfying the relation $u=\mathcal{T}_1(\lambda)g$. This equivalence holds true for all $\lambda\in\mathbb{S}_\varepsilon$.*

Proof. The operator $(-\Delta_{\mathbb{R}^n}-\lambda)^{-1}:L_2(\mathbb{R}^n;\Omega^\beta)\to W_2^2(\mathbb{R}^n)$ is bounded and holomorphic w.r.t. $\lambda\in\mathbb{S}_\varepsilon$ that can be established by analogy with the proof of Lemma 3.1 in [1]. Since $(\mathcal{H}_\Omega-\mathrm{i})^{-1}\mathcal{L}$ is a bounded operator in $W_{2,0}^2(\Omega^\beta)$, we conclude that the mapping $g\mapsto w$ is a bounded operator from $L_2(\mathbb{R}^n;\Omega^\beta)$ into $W_{2,0}^2(\Omega^\beta)$ being holomorphic w.r.t. $\lambda\in\mathbb{S}_\varepsilon$. Thus, the operator $\mathcal{T}_1(\lambda):L_2(\Omega^\beta)\to W_2^2(\mathbb{R}^n)$ is bounded and holomorphic w.r.t. $\lambda\in\mathbb{S}_\varepsilon$. This fact and the definition of $\mathcal{T}_2$ imply that this operator is bounded and holomorphic w.r.t. $\lambda\in\mathbb{S}_\varepsilon$ as an operator in $L_2(\Omega^\beta)$.

Let g solve the equation (3.5); as it was shown above in this case the function u defined by (3.3) is a solution to the equation (3.1). Suppose now that u solves (3.1). By direct calculations one can check that the corresponding v, w and g are given by the formulas

$$w:=(\Delta_{\Omega^\beta}^D+\mathrm{i})^{-1}\mathcal{L}u,\quad v:=u-\chi_\Omega w,\quad g=\mathcal{T}_1^{-1}(\lambda)u:=(-\Delta-\lambda)v, \tag{3.6}$$

where $\Delta_{\Omega^\beta}^D$ is the Dirichlet Laplacian in Ω^β. ∎

Lemma 3.2. *The operator* $(\mathrm{I}+\mathcal{T}_2)^{-1}$ *is bounded and meromorphic on* $\lambda \in \mathbb{S}_\varepsilon$. *The poles of this operator are simple and coincide with the isolated eigenvalues of* $\mathcal{H}_\mathcal{L}$. *For* λ *close to a p-multiple eigenvalue* λ_* *of* $\mathcal{H}_\mathcal{L}$ *the representation*

$$(\mathrm{I}+\mathcal{T}_2(\lambda))^{-1} = -\sum_{i=1}^{p} \frac{\phi_i(\cdot,\psi_i)_{L_2(\Omega^\beta)}}{\lambda-\lambda_*} + \mathcal{T}_3(\lambda) \tag{3.7}$$

holds true. Here ψ_i *are the eigenfunctions associated with* λ_* *and orthonormalized in* $L_2(\mathbb{R}^n)$, $\phi_i := \mathcal{T}_1^{-1}(\lambda_*)\psi_i$, *and the operator* $\mathcal{T}_3 : L_2(\Omega^\beta) \to L_2(\Omega^\beta)$ *is bounded and holomorphic w.r.t.* λ *close to* λ_* *as an operator in* $L_2(\Omega^\beta)$. *The equation (3.5) with* $\lambda=\lambda_*$ *is solvable if and only if*

$$(f,\psi_i)_{L_2(\Omega^\beta)} = 0, \quad i=1,\ldots,p, \tag{3.8}$$

and the solution reads as follows

$$g = \mathcal{T}_3(\lambda_*)f + \sum_{j=1}^{m} c_i\phi_i, \tag{3.9}$$

where c_i *are arbitrary constants.*

Proof. It follows from (3.3), (3.4) that $(\mathcal{H}_\mathcal{L}-\lambda)\mathcal{T}_1(\lambda) = \mathrm{I}+\mathcal{T}_2(\lambda)$. Therefore,

$$(\mathcal{H}_\mathcal{L}-\lambda)^{-1} = \mathcal{T}_1(\lambda)(\mathrm{I}+\mathcal{T}_2(\lambda))^{-1}, \quad (\mathrm{I}+\mathcal{T}_2(\lambda))^{-1} = \mathcal{T}_1^{-1}(\lambda)(\mathcal{H}_\mathcal{L}-\lambda)^{-1}, \tag{3.10}$$

where the operator $\mathcal{T}_1^{-1}(\lambda)$ is defined by the formulas (3.6). By analogy with the proof of Lemma 3.1 in [1] one can show that the operator $(\mathcal{H}_\mathcal{L}-\lambda)^{-1} : L_2(\mathbb{R}^n;\Omega^\beta) \to W_2^2(\mathbb{R}^n)$ is meromorphic on $\lambda \in \mathbb{S}_\varepsilon$, its poles coincide with the isolated eigenvalues of $\mathcal{H}_\mathcal{L}$, and for λ close to λ_* the representation

$$(\mathcal{H}_\mathcal{L}-\lambda)^{-1} = -\sum_{i=1}^{p} \frac{\psi_i(\cdot,\psi_i)_{L_2(\mathbb{R}^n)}}{\lambda-\lambda_*} + \mathcal{T}_4(\lambda) \tag{3.11}$$

holds true, where the operator $\mathcal{T}_4(\lambda) : L_2(\mathbb{R}^n) \to W_2^2(\mathbb{R}^n)$ is bounded and holomorphic w.r.t. λ close to λ_*. Hence, in view of (3.10), (3.11), and (3.6), the operator $(\mathrm{I}+\mathcal{T}_2)^{-1}$ is meromorphic on $\lambda \in \mathbb{S}_\varepsilon$, the poles of this operator are simple and coincide with the isolated eigenvalues of $H_\mathcal{L}$, and the representation (3.7) holds true. As it also follows from (3.11), the equation (3.1) with $\lambda=\lambda_*$ is solvable if and only if the relations (3.8) are valid, and the solution of (3.1) with $\lambda=\lambda_*$ is given by the formula $u = \mathcal{T}_4(\lambda_*)f + \sum_{j=1}^{p} c_i\psi_i$, where c_i are arbitrary constants. Employing now Lemma 3.1, we conclude that the relations (3.8) are the solvability conditions for the equation (3.6) with $\lambda=\lambda_*$. Thus, the solution of this equation is defined uniquely up to a linear combination of the functions ϕ_i, $i=1,\ldots,m$. The formula (3.9) is valid since for the functions f satisfying (3.8) the identity $\big(\mathrm{I}+\mathcal{T}_2(\lambda_*)\big)\mathcal{T}_3(\lambda_*)f = f$ holds true due to (3.7). $\square$

Let $\widetilde{\Omega} \subset \mathbb{R}^n$ be a bounded domain with infinitely differentiable boundary, and $\widetilde{X} \in \mathbb{R}^n$ be a point. Suppose that $l := |\widetilde{X}|$ is a large parameter. We define the operator $\mathcal{T}_5(\lambda, \widetilde{X}) : L_2(\mathbb{R}^n; \Omega^\beta) \to W_2^2(\widetilde{\Omega})$ as follows

$$\mathcal{T}_5(\lambda, \widetilde{X}) := \mathcal{S}(\widetilde{X})(-\Delta_{\mathbb{R}^n} - \lambda)^{-1}.$$

Lemma 3.3. *The operator $\mathcal{T}_5$ is bounded and holomorphic on $\lambda \in \mathbb{S}_\varepsilon$. For any compact set $\mathbb{K} \in \mathbb{S}_\varepsilon$ the estimates*

$$\left\| \frac{\partial^i \mathcal{T}_5}{\partial \lambda^i} \right\| \leqslant C l^{-\frac{n-2i-1}{2}} \mathrm{e}^{-l\sqrt{-\lambda}}, \quad i = 0, 1, \tag{3.12}$$

hold true, where the constant C is independent on $\widetilde{X}$ and $\lambda \in \mathbb{K}$.

Proof. As it was said in the proof of Lemma 3.2 the operator $(-\Delta_{\mathbb{R}^n} - \lambda)^{-1}$: $L_2(\mathbb{R}^n; \Omega^\beta) \to W_2^2(\mathbb{R}^n)$ is bounded and holomorphic w.r.t. $\lambda \in \mathbb{S}_\varepsilon$. Therefore, the same is true for the operator $\mathcal{T}_5$. The estimates (3.12) follow from the asymptotics

$$G_n(t, \lambda) = -\frac{(\sqrt[4]{-\lambda})^{n-3}}{2^{(n+1)/2}\pi^{(n-1)/2}} t^{-(n-1)/2} \mathrm{e}^{-t\sqrt{-\lambda}} \Big(1 + \mathcal{O}\big(|\lambda|^{-1/2} t^{-1}\big)\Big), \tag{3.13}$$

as $t \to +\infty$, $\lambda \in \mathbb{S}_\varepsilon$; this formula can be differentiated w.r.t. λ. $\square$

4 Equation for the eigenelements of $\mathcal{H}_X$

In this section we will obtain the equation for the eigenvalues and the eigenfunctions of the operator $\mathcal{H}_X$ and will solve this equation explicitly.

By $\mathcal{T}_j^{(i)}$, $\mathcal{T}_j^{(X)}$, we denote the operators $\mathcal{T}_j$ from the previous section corresponding to $\mathcal{L} = \mathcal{L}_j$, $\Omega = \Omega_j$, $\mathcal{L} = \mathcal{L}_X$, $\Omega = \Omega_X$. Let us study the structure of the operator $\mathcal{T}_2^{(X)}$ in more details.

Given $g \in L_2(\Omega_X^\beta)$, due to (3.2) we have

$$\begin{aligned} v_X(x, \lambda) &= \int\limits_{\Omega_X^\beta} G_n(|x-t|, \lambda) g(t)\,\mathrm{d}t = \sum_{i=1}^m \int\limits_{\Omega_i^\beta + \{X_i\}} G_n(|x-t|, \lambda) g(t)\,\mathrm{d}t \\ &= \sum_{i=1}^m \int\limits_{\Omega_i^\beta} G_n(|x - X_i - t|, \lambda) g_i(t)\,\mathrm{d}t = \sum_{i=1}^m \big(\mathcal{S}(-X_i) v_i\big)(x, \lambda), \end{aligned} \tag{4.1}$$

$$g_i(t) := g(X_i + t), \quad v_i(x) := \int\limits_{\Omega_i^\beta} G_n(|x-t|, \lambda) g_i(t)\,\mathrm{d}t.$$

Now we apply the operator $\mathcal{L}_X$ to the function v_X and obtain:

$$\mathcal{L}_X v_X = \sum_{i=1}^m \mathcal{S}(-X_i) \mathcal{L}_i \left(v_i + \sum_{\substack{j=1 \\ j \neq i}}^m \mathcal{S}(X_{i,j}) v_j \right) = \sum_{i=1}^m \mathcal{S}(-X_i) \mathcal{L}_i \left(v_i + \widetilde{v}_i \right), \tag{4.2}$$

$$\widetilde{v}_i := \sum_{\substack{j=1\\ j\neq i}}^{m} \mathcal{S}(X_{i,j})v_j = \sum_{\substack{j=1\\ j\neq i}}^{m} \mathcal{T}_5(\lambda, X_{i,j})g_j.$$

We introduce the functions

$$\begin{aligned}
&w_i := -(\mathcal{H}_{\Omega_i} - \mathrm{i})^{-1}\mathcal{L}_i v_i, \quad \widetilde{w}_i := -(\mathcal{H}_{\Omega_i} - \mathrm{i})^{-1}\mathcal{L}_i \widetilde{v}_i,\\
&w_X := W_X + \widetilde{W}_X, \quad W_X := \sum_{i=1}^{m} \mathcal{S}(-X_i)w_i, \quad \widetilde{W}_X := \sum_{i=1}^{m} \mathcal{S}(-X_i)\widetilde{w}_i.
\end{aligned}$$

It is obvious that $w_X, W_X, \widetilde{W}_X \in W^2_{2,0}(\Omega^\beta_X)$. Since $\mathcal{L}_X w_X = \sum\limits_{i=1}^{m} \mathcal{S}(-X_i)\mathcal{L}_i(w_i + \widetilde{w}_i)$, we obtain

$$\begin{aligned}
(\mathcal{H}_{\Omega_X} - \mathrm{i})w_X &= \sum_{i=1}^{m} \Big(-(\Delta + \mathrm{i})\mathcal{S}(-X_i) + \mathcal{S}(-X_i)\mathcal{L}_i\Big)(w_i + \widetilde{w}_i)\\
&= -\sum_{i=1}^{m} \mathcal{S}(-X_i)\mathcal{L}_i(v_i + \widetilde{v}_i) = -\mathcal{L}_X v_X,\\
w_X &= -(\mathcal{H}_{\Omega_X} - \mathrm{i})^{-1}\mathcal{L}_X v_X.
\end{aligned}$$

We define the cut-off function $\chi_{\Omega_X} := \sum_{i=1}^{m} \mathcal{S}(-X_i)\chi_{\Omega_i}$, where the function χ_{Ω_i} corresponds to the operator $\mathcal{T}_1^{(i)}$. In this case the operator $\mathcal{T}_1^{(X)}$ reads as follows:

$$\begin{aligned}
\mathcal{T}_1^{(X)} g &= \sum_{i=1}^{m} \mathcal{S}(-X_i)v_i + \sum_{i=1}^{m} \mathcal{S}(-X_i)\chi_{\Omega_i}(w_i + \widetilde{w}_i)\\
&= \sum_{i=1}^{m} \mathcal{S}(-X_i)(v_i + \chi_{\Omega_i}w_i + \chi_{\Omega_i}\widetilde{w}_i) = \sum_{i=1}^{m} \mathcal{S}(-X_i)(\mathcal{T}_1^{(i)} g_i + \chi_{\Omega_i}\widetilde{w}_i).
\end{aligned}$$

Therefore,

$$\mathcal{T}_2^{(X)}(\lambda, X)g = \sum_{i=1}^{m} \mathcal{S}(-X_i)\mathcal{T}_2^{(i)}(\lambda)g_i + \sum_{i=1}^{m} \mathcal{S}(-X_i) \sum_{\substack{j=1\\ j\neq i}}^{m} \mathcal{T}_6^{(i,j)}(\lambda)g_j, \tag{4.3}$$

$$\mathcal{T}_6^{(i,j)}(\lambda) := (2\nabla\chi_{\Omega_i}\cdot\nabla + (\Delta\chi_{\Omega_i} + (\lambda - \mathrm{i})\chi_{\Omega_i}))\,(\mathcal{H}_{\Omega_i} - \mathrm{i})^{-1}\mathcal{L}_i\mathcal{T}_5(\lambda, X_{i,j}).$$

Lemma 4.1. *The operators $\mathcal{T}_6^{(i,j)} : L_2(\mathbb{R}^n; \Omega^\beta_j) \to L_2(\Omega^\beta_i)$ are bounded and holomorphic w.r.t. $\lambda \in \mathbb{S}_\varepsilon$. The relation*

$$\mathcal{T}_6^{(i,j)}(\lambda) = \mathcal{L}_i\mathcal{T}_5(\lambda, X_{i,j}) + (\Delta - \mathcal{L}_i + \lambda)\chi_{\Omega_i}(\mathcal{H}_{\Omega_i} - \mathrm{i})^{-1}\mathcal{L}_i\mathcal{T}_5(\lambda, X_{i,j}) \tag{4.4}$$

is valid. For each compact set $\mathbb{K} \subset \mathbb{S}_\varepsilon$ the estimates

$$\left\| \frac{\partial^k \mathcal{T}_6^{(i,j)}}{\partial\lambda^k} \right\| \leqslant C l_{i,j}^{-\frac{n-2k-1}{2}} \mathrm{e}^{-l_{i,j}\sqrt{-\lambda}}, \quad k = 0, 1,$$

hold true, where the constant C is independent on $l_{i,j}$ and $\lambda \in \mathbb{K}$.

The statement of the lemma follows from the definition of $\mathcal{T}_6^{(i,j)}$ and Lemma 3.3.

According to Lemma 3.1, the eigenvalues of the operator $\mathcal{H}_X$ are numbers for which the equation (3.5) with $\mathcal{T}_2 = \mathcal{T}_2^{(X)}$ and $f = 0$ has a nontrivial solution. Let g_X be a solution to this equation. Since $g_X = \sum\limits_{i=1}^{m} \mathcal{S}(-X_i)g_i$, due to (4.3) we conclude that the equation (3.5) for g_X can be rewritten as

$$\sum_{i=1}^{m} \mathcal{S}(-X_i) \left(g_i + \mathcal{T}_2^{(i)}(\lambda)g_i + \sum_{\substack{j=1 \\ j\neq i}}^{m} \mathcal{T}_6^{(i,j)}(\lambda)g_j \right) = 0.$$

Each term in this equation has a compact support and these supports do not intersect if $l_{i,j}$ are large enough. Thus, the obtained equation is equivalent to

$$g_i + \mathcal{T}_2^{(i)}(\lambda)g_i + \sum_{\substack{j=1 \\ j\neq i}}^{m} \mathcal{T}_6^{(i,j)}(\lambda)g_j = 0, \quad i,j = 1,\ldots,m. \tag{4.5}$$

We introduce two operators in the space $\boldsymbol{L} := \bigoplus\limits_{i=1}^{m} L_2(\mathbb{R}^n;\Omega_i^\beta)$,

$$\mathcal{T}_7(\lambda)\boldsymbol{g} := \big(\mathcal{T}_2^{(1)}(\lambda)g_1,\ldots,\mathcal{T}_2^{(m)}(\lambda)g_m\big),$$

$$\mathcal{T}_8(\lambda,X)\boldsymbol{g} := \left(\sum_{\substack{j=1 \\ j\neq 1}}^{m} \mathcal{T}_6^{(1,j)}(\lambda)g_j, \ldots, \sum_{\substack{j=1 \\ j\neq m}}^{m} \mathcal{T}_6^{(m,j)}(\lambda)g_j \right),$$

where $\boldsymbol{g} := (g_1,\ldots,g_m) \in \boldsymbol{L}$. Employing these operators, we can rewrite the equation (4.5) as follows:

$$\boldsymbol{g} + \mathcal{T}_7(\lambda)\boldsymbol{g} + \mathcal{T}_8(\lambda,X)\boldsymbol{g} = 0. \tag{4.6}$$

Let $\lambda_* \in \sigma_*$ be $(p_1+\ldots+p_m)$-multiple, and $\psi_{i,j}$, $i=1,\ldots,m$, $j=1,\ldots,p_i$, be the associated eigenfunctions of $\mathcal{H}_i$ orthonormalized in $L_2(\mathbb{R}^n)$. We denote $p := p_1+\ldots+p_m$,

$$\begin{aligned}
&\boldsymbol{\phi}_{\alpha_1+j} := (\phi_{1,j},0,0,\ldots,0) \in \boldsymbol{L}, && \mathcal{T}_9^{(\alpha_1+j)}\boldsymbol{g} := (g_1,\phi_{1,j})_{L_2(\Omega_1^\beta)}, && j=1,\ldots,p_1,\\
&\boldsymbol{\phi}_{\alpha_2+j} := (0,\phi_{2,j},0,\ldots,0) \in \boldsymbol{L}, && \mathcal{T}_9^{(\alpha_2+j)}\boldsymbol{g} := (g_2,\phi_{2,j})_{L_2(\Omega_2\beta)}, && j=1,\ldots,p_2,\\
&\qquad\ldots && \qquad\ldots && \qquad\ldots\\
&\boldsymbol{\phi}_{\alpha_m+j} := (0,0,\ldots,0,\phi_{m,j}) \in \boldsymbol{L}, && \mathcal{T}_9^{(\alpha_m+j)}\boldsymbol{g} := (g_m,\phi_{m,j})_{L_2(\Omega_m^\beta)}, && j=1,\ldots,p_m.
\end{aligned}$$

Here $\phi_{i,j} := \big(\mathcal{T}_1(\lambda_*)\big)^{-1}\psi_{i,j}$. Lemmas 3.2, 4.1 yield

Lemma 4.2. *The operator $\mathcal{T}_8$ is bounded and holomorphic w.r.t. $\lambda \in \mathbb{S}_\varepsilon$. For each compact set $\mathbb{K} \in \mathbb{S}_\varepsilon$ the uniform in $\lambda \in \mathbb{K}$ and large $l_{i,j}$ estimates*

$$\left\| \frac{\partial^i \mathcal{T}_8}{\partial \lambda^i} \right\| \leqslant C l_X^{-\frac{n-2i-1}{2}} \mathrm{e}^{-l_X\sqrt{-\lambda}}, \quad i = 0, 1, \tag{4.7}$$

are valid. The operator $\mathcal{T}_7$ is bounded and meromorphic w.r.t. $\lambda \in \mathbb{S}_\varepsilon$. The set of its poles coincide with σ_*. For any λ close to $(p_1 + \ldots + p_m)$-multiple $\lambda_* \in \sigma_*$ the representation

$$\big(\mathrm{I} + \mathcal{T}_7(\lambda)\big)^{-1} = -\sum_{i=1}^{p} \frac{\boldsymbol{\phi}_i \mathcal{T}_9^{(i)}}{\lambda - \lambda_*} + \mathcal{T}_{10}(\lambda), \tag{4.8}$$

holds true, where the j-th component of the vector $\mathcal{T}_{10}(\lambda)\boldsymbol{g}$ is $\mathcal{T}_3^{(j)} g_j$ if $p_j \neq 0$ and $\big(\mathrm{I} + \mathcal{T}_7^{(j)}(\lambda)\big)^{-1} g_j$ if $p_j = 0$. The operator $\mathcal{T}_{10} : \boldsymbol{L} \to \boldsymbol{L}$ is bounded and holomorphic w.r.t. λ close to λ_*. The equation $\big(\mathrm{I} + \mathcal{T}_7(\lambda_*)\big)\boldsymbol{g} = \boldsymbol{f}$ is solvable if and only if $\mathcal{T}_9^{(i)}\boldsymbol{f} = 0$, $i = 1, \ldots, m$. The solution of this equation is given by $\boldsymbol{g} = \mathcal{T}_{10}(\lambda_*)\boldsymbol{f} + \sum\limits_{i=1}^{p} c_i \boldsymbol{\phi}_i$, where c_i are some constants.

Lemma 4.3. *Each isolated eigenvalue of $\mathcal{H}_X$ converges to zero or to $\lambda_* \in \sigma_*$ as $l_X \to +\infty$.*

Proof. Using (1.2), (1.3), for each $u \in W_2^1(\mathbb{R}^n)$ we obtain

$$(\mathcal{H}_X u, u)_{L_2(\mathbb{R}^n)} \geqslant \|\nabla u\|^2_{L_2(\mathbb{R}^n)} - c_0 \|\nabla u\|^2_{L_2(\Omega_X)} - c_1 \|u\|^2_{L_2(\Omega_X)} \geqslant -c_1 \|u\|^2_{L_2(\mathbb{R}^n)},$$

which implies that $\sigma_{\mathrm{disc}}(\mathcal{H}_X) \subset [-c_1, 0)$. We define $\mathbb{K}_\varepsilon := [-c_1, -\varepsilon) \setminus \bigcup\limits_{\lambda \in \sigma_*} (\lambda - \varepsilon, \lambda + \varepsilon)$. This set obeys the hypothesis of Lemma 3.2, and due to (4.7) the norm of $\mathcal{T}_8$ is exponentially small as $\lambda \in \mathbb{K}_\varepsilon$ and $l_X \to +\infty$. In accordance with Lemma 4.2, the operator $\mathrm{I} + \mathcal{T}_7(\lambda)$ is boundedly invertible as $\lambda \in \mathbb{K}_\varepsilon$. Therefore, the operator $\mathrm{I} + \mathcal{T}_7(\lambda) + T_8(\lambda, X)$ is boundedly invertible as $\lambda \in \mathbb{K}_\varepsilon$ if l_X is large enough. Thus, the equation (4.6) has no nontrivial solution as $\lambda \in \mathbb{K}_\varepsilon$ if l_X is large enough, and by Lemma 3.1 we conclude that the set $\mathbb{K}_\varepsilon$ contains no eigenvalues of $\mathcal{H}_X$ if l_X is large enough. The number ε being arbitrary completes the proof. ☐

Let $\lambda_* \in \sigma_*$ be $(p_1 + \ldots + p_m)$-multiple; we are going to find non-trivial solutions of (4.6) for λ close to λ_*.

Assume first that $\lambda \neq \lambda_*$. We apply the operator $(\mathrm{I} + \mathcal{T}_7)^{-1}$ to this equation and then substitute the representation (4.8) into the relation obtained. This procedure yields

$$\boldsymbol{g} - \sum_{i=1}^{p} \frac{\phi_i \mathcal{T}_9^{(i)} \mathcal{T}_8(\lambda, X)\boldsymbol{g}}{\lambda - \lambda_*} + \mathcal{T}_{10}(\lambda)\mathcal{T}_8(\lambda, X)\boldsymbol{g} = 0. \tag{4.9}$$

In view of (4.7) the operator $\mathcal{T}_{10}(\lambda)\mathcal{T}_8(\lambda, X)$ is small if l_X is large enough. Thus, the operator $\big(\mathrm{I} + \mathcal{T}_{10}(\lambda)\mathcal{T}_8(\lambda, X)\big)^{-1}$ is well-defined and bounded. We apply now

this operator to the equation (4.9) and arrive at

$$\boldsymbol{g}-\sum_{i=1}^{p}\frac{\mathcal{T}_9^{(i)}\mathcal{T}_8(\lambda,X)\boldsymbol{g}}{\lambda-\lambda_*}\boldsymbol{\Phi}_i=0, \tag{4.10}$$

where $\boldsymbol{\Phi}_i(\cdot,\lambda,X):=\big(\mathrm{I}+\mathcal{T}_{10}(\lambda)\mathcal{T}_8(\lambda,X)\big)^{-1}\boldsymbol{\phi}_i$. Hence,

$$\boldsymbol{g}=\sum_{i=1}^{p}\kappa_i\boldsymbol{\Phi}_i, \tag{4.11}$$

where κ_i are some numbers to be found. We substitute now this identity into (4.10) and obtain

$$\sum_{i=1}^{p}\boldsymbol{\Phi}_i\left(\kappa_i-\sum_{j=1}^{p}A_{ij}\kappa_j\right)=0, \tag{4.12}$$

$$A_{ij}=A_{ij}(\lambda,X):=\mathcal{T}_9^{(i)}\mathcal{T}_8(\lambda,X)\boldsymbol{\Phi}_j(\cdot,\lambda,X).$$

The estimates (4.7) imply that

$$\boldsymbol{\Phi}_i=\boldsymbol{\phi}_i+\mathcal{O}(l_X^{-\frac{n-1}{2}}\mathrm{e}^{-l_X\sqrt{-\lambda}}),\quad l_X\to+\infty. \tag{4.13}$$

Since the vectors $\boldsymbol{\phi}_i$ are linear independent, due to these relations the same is true for $\boldsymbol{\Phi}_i$. Thus, the equation (4.12) is equivalent to the system of linear equations

$$\big((\lambda-\lambda_*)\mathrm{E}-\mathrm{A}(\lambda,X)\big)\boldsymbol{\kappa}=0, \tag{4.14}$$

$$\boldsymbol{\kappa}:=\begin{pmatrix}\kappa_1\\ \vdots\\ \kappa_p\end{pmatrix},\quad \mathrm{A}(\lambda,X):=\begin{pmatrix}A_{11}(\lambda,X) & \dots & A_{1p}(\lambda,X)\\ \vdots & & \vdots\\ A_{p1}(\lambda,X) & \dots & A_{pp}(\lambda,X)\end{pmatrix}, \tag{4.15}$$

where E is the identity matrix. The corresponding solution of the equation (4.6) is given by (4.11). Since the vectors $\boldsymbol{\Phi}_i$ are linear independent, this solution is non-zero if and only if $\boldsymbol{\kappa}\neq 0$. The criterion of the existence of nontrivial solution to (4.14) is

$$\det\big((\lambda-\lambda_*)\mathrm{E}-\mathrm{A}(\lambda,X)\big)=0. \tag{4.16}$$

Therefore, the number $\lambda\neq\lambda_*$ converging to λ_* as $l_X\to+\infty$ is an eigenvalue the operator $\mathcal{H}_X$ if and only if it is a root of the obtained equation. The multiplicity of this eigenvalue equals to the number of linear independent solutions of the corresponding system (4.14). Let us prove that the same is true if $\lambda=\lambda_*$.

Consider the equation (4.6) with $\lambda=\lambda_*$. If we treat $\mathcal{T}_8(\lambda_*,X)\boldsymbol{g}$ as a right-hand side, according to Lemma 4.2 this equation is solvable if and only if

$$\mathcal{T}_9^{(i)}\mathcal{T}_8(\lambda_*,X)\boldsymbol{g}=0,\quad i=1,\dots,m, \tag{4.17}$$

and the solution is given by

$$\boldsymbol{g} = -\mathcal{T}_{10}(\lambda_*)\mathcal{T}_8(\lambda_*, X)g + \sum_{i=1}^{p} \kappa_i \boldsymbol{\phi}_i,$$

where κ_i are some constants. Now we apply the operator $\big(\mathrm{I} + \mathcal{T}_{10}(\lambda_*)\mathcal{T}_8(\lambda_*, X)\big)^{-1}$ to this identity and arrive at the formula (4.11) with $\lambda = \lambda_*$. We substitute this formula into (4.17) and obtain the system (4.14) with $\lambda = \lambda_*$. The vector $\boldsymbol{g}$ is non-zero if and only if $\boldsymbol{\kappa} \neq 0$; this leads us to the equation (4.16) with $\lambda = \lambda_*$.

It is convenient to summarize the obtained results in

Lemma 4.4. *Let $\lambda_* \in \sigma_*$ be $(p_1 + \ldots + p_m)$-multiple. A number $\lambda \xrightarrow[l_X \to +\infty]{} \lambda_*$ is an eigenvalue the $\mathcal{H}_X$ if and only if it is a root of (4.16). The multiplicity of this eigenvalue equals to the number of linear independent solutions of the corresponding system (4.14).*

5 Proof of Theorems 1.2–1.4

In view of Lemmas 4.3, 4.4 we will complete the proof of Theorem 1.2, if we prove that total number of non-trivial solutions to (4.14) associated with the roots of (4.16) equals p.

Throughout this section we assume that $\lambda_* \in \sigma_*$ is $(p_1+\ldots+p_m)$-multiple and λ belongs to a small neighbourhood of λ_*. We denote $\mathrm{B}(\lambda, X) := (\lambda - \lambda_*)\mathrm{E} - \mathrm{A}(\lambda, X)$, $F(\lambda, X) := \det \mathrm{B}(\lambda, X)$.

Lemma 5.1. *In the vicinity of λ_* the function $F(\lambda, X)$ has exactly p zeroes counting their orders. These zeroes converge to λ_* as $l_X \to +\infty$.*

Proof. The definition of the functions $A_{i,j}$ and Lemma 4.2 imply that these functions are holomorphic w.r.t. λ and satisfy the estimates

$$\left| \frac{\partial^k A_{ij}}{\partial \lambda^k}(\lambda, l) \right| \leqslant C l_X^{-\frac{n-2k-1}{2}} \mathrm{e}^{-l_X \sqrt{-\lambda}}, \quad k = 0, 1.$$

It is clear that

$$F(\lambda, X) = (\lambda - \lambda_*)^p + \sum_{i=0}^{p-1} P_i(\lambda, X)(\lambda - \lambda_*)^i,$$

where the functions P_i are holomorphic w.r.t. λ and obey the uniform in estimate

$$|P_i(\lambda, X)| \leqslant C l_X^{-\frac{(p-i)(n-1)}{2}} \mathrm{e}^{-(p-i) l_X \sqrt{-\lambda}}.$$

For a sufficiently small fixed $\varepsilon > 0$ this estimate yields

$$\left| \sum_{i=0}^{p-1} P_i(\lambda, X)(\lambda - \lambda_*)^i \right| < |\lambda - \lambda_*|^p \quad \text{as} \quad |\lambda - \lambda_*| = \varepsilon,$$

if l_X is large enough. Hence, by Rouche theorem the function $F(\lambda, X)$ has the same number of zeroes (counting orders) inside the disk $\{\lambda : |\lambda - \lambda_*| < \varepsilon\}$ as the function $\lambda \mapsto (\lambda - \lambda_*)^p$ does. The number ε being arbitrary completes the proof. $\square$

Lemma 5.2. *Suppose that $\lambda_1(X)$ and $\lambda_2(X)$ are different roots of the equation (4.16), and $\boldsymbol{\kappa}_1(X)$ and $\boldsymbol{\kappa}_2(X)$ are the associated non-trivial solutions to the system (4.14) normalized by the condition*

$$\|\boldsymbol{\kappa}_i\|_{\mathbb{C}^p} = 1.$$

Then

$$\big(\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2\big)_{\mathbb{C}^p} = \mathcal{O}(l_X^{\frac{n-1}{2}} \mathrm{e}^{-l_X\sqrt{-\lambda}}), \quad l_X \to +\infty.$$

Proof. We indicate by $\boldsymbol{g}_j$ the solutions of the equation (4.6) associated with λ_j; these solutions are given by (4.11). Due to Lemma 3.3 the functions $\widetilde{v}_i$ and $\widetilde{w}_i$ corresponding to each of the vectors $\boldsymbol{g}_j$ satisfy the estimates

$$\|\mathcal{L}_i\widetilde{v}_i\|_{L_2(\Omega_i^\beta)} = \mathcal{O}(l_X^{-\frac{(n-1)}{2}} \mathrm{e}^{-l_X\sqrt{-\lambda_j}}), \quad \|\widetilde{w}_i\|_{W_2^2(\Omega_i^\beta)} = \mathcal{O}(l_X^{-\frac{(n-1)}{2}} \mathrm{e}^{-l_X\sqrt{-\lambda_j}}),$$

as $l_X \to +\infty$. Moreover, if follows from (4.13) that

$$\boldsymbol{g}_i = \sum_{j=1}^{p} \kappa_j^{(i)} \boldsymbol{\phi}_j + \mathcal{O}(l_X^{-\frac{n-1}{2}} \mathrm{e}^{-l_X\sqrt{-\lambda_i}}),$$

where $\kappa_j^{(i)}$ are the components of the vectors $\boldsymbol{\kappa}_i$. In view of the relation obtained and (4.1), (4.2) we infer that the eigenfunctions $\psi_i(x, X)$ associated with λ_i satisfy the asymptotic formulas:

$$\psi_i = \sum_{j=1}^{m} \mathcal{S}(-X_j) \sum_{q=1}^{p_j} \kappa^{(i)}_{\alpha_j+q}\psi_{j,q} + \mathcal{O}\big(l_X^{-\frac{n-1}{2}} \mathrm{e}^{-l_X\sqrt{-\lambda_j}}\big),$$

where, we remind, $\psi_{i,j}$, $j = 1, \ldots, p_i$, are the eigenfunctions of $\mathcal{H}_i$ associated with λ_* and orthonormalized in $L_2(\mathbb{R}^n)$. Since the operator $\mathcal{H}_X$ is self-adjoint, the eigenfunctions ψ_i are orthogonal in $L_2(\mathbb{R}^n)$. Together with the established asymptotic representations for ψ_i it implies

$$\begin{aligned} 0 = (\psi_1, \psi_2)_{L_2(\mathbb{R}^n)} = &\sum_{i,j=1}^{m} \sum_{q=1}^{p_j} \sum_{r=1}^{p_i} \kappa^{(1)}_{\alpha_j+q}\overline{\kappa}^{(2)}_{\alpha_i+r} \Big(\mathcal{S}(-X_j)\psi_{j,q}, \mathcal{S}(-X_i)\psi_{i,r}\Big)_{L_2(\mathbb{R}^n)} \\ &+ \mathcal{O}\big(l_X^{-\frac{n-1}{2}} \mathrm{e}^{-l_X\sqrt{-\max\{\lambda_1,\lambda_2\}}}\big). \end{aligned} \tag{5.1}$$

It is clear that

$$\Big(\mathcal{S}(-X_j)\psi_{j,q}, \mathcal{S}(-X_j)\psi_{j,r}\Big)_{L_2(\mathbb{R}^n)} = (\psi_{j,q}, \psi_{j,r})_{L_2(\mathbb{R}^n)} = \begin{cases} 1, & q = r, \\ 0, & q \neq r, \end{cases}$$

and for $i \neq j$

$$
\begin{aligned}
&\Big(\mathcal{S}(-X_j)\psi_{j,q}, \mathcal{S}(-X_i)\psi_{i,r}\Big)_{L_2(\mathbb{R}^n)} = \Big(\mathcal{S}(X_{i,j})\psi_{j,q}, \psi_{i,r}\Big)_{L_2(\mathbb{R}^n)} \\
&= \Big(\mathcal{S}(X_{i,j})\psi_{j,q}, \psi_{i,r}\Big)_{L_2\left(\Omega_i^{l_{i,j}/2}\right)} + \Big(\mathcal{S}(X_{i,j})\psi_{j,q}, \psi_{i,r}\Big)_{L_2\left(\mathbb{R}^n\setminus\Omega_i^{l_{i,j}/2}\right)} = \mathcal{O}\left(l_{i,j}^{\frac{n-1}{2}}\mathrm{e}^{-l_{i,j}\sqrt{-\lambda_*}}\right),
\end{aligned}
$$

Here we have used that due to (3.13)

$$
\psi_{i,j} = C_{i,j}|x|^{-(n-1)/2}\mathrm{e}^{-|x|\sqrt{-\lambda}}\Big(1 + \mathcal{O}\big(|x|^{-1}\big)\Big), \quad |x| \to +\infty,
$$

where $C_{i,j}$ are some constants. Substituting the obtained relations into (5.1), we arrive at the statement of the lemma. $\square$

Let $\lambda(X) \xrightarrow[l_X\to+\infty]{} \lambda_*$ be a root of the equation (4.16). Without loss of generality we assume that the corresponding solutions of (4.14) are orthonormalized in $\mathbb{C}^p$. Consider the set of all such solutions to (4.14) associated with all roots of (4.16) converging to λ_* as $l_X \to +\infty$, and denote these vectors as $\boldsymbol{\kappa}_i = \boldsymbol{\kappa}_i(X)$, $i = 1,\dots,q$. In view of Lemma 5.2 the vectors $\boldsymbol{\kappa}_i$ satisfy the formulas (1.5).

Lemma 5.3. *Let $\lambda(X) \xrightarrow[l_X\to+\infty]{} \lambda_*$ be a root of the equation (4.16) and $\boldsymbol{\kappa}_i$, $i = N,\dots,N+q$, $q \geqslant 0$, be the associated solutions to (4.14). Then the representation*

$$
\mathrm{B}^{-1}(\lambda, X) = \sum_{i=N}^{N+q} \frac{\mathcal{T}_{11}^{(i)}(X)}{\lambda - \lambda(X)}\boldsymbol{\kappa}_i(X) + \mathrm{B}_0(\lambda, X)
$$

is valid for all λ close to $\lambda(X)$. Here $\mathcal{T}_{11}^{(i)} : \mathbb{C}^p \to \mathbb{C}$ are some functionals, while the matrix $\mathrm{B}_0(\lambda, X)$ is holomorphic w.r.t. λ in a neighbourhood of $\lambda(X)$.

Proof. The matrix B is meromorphic and its inverse thus has a pole at $\lambda(X)$. By analogy with the relations (5.7), (5.8) in [1] one can show that the residue at this pole is of the form $\sum_{i=N}^{N+q}\boldsymbol{\kappa}_i(X)\mathcal{T}_{11}^{(i)}(X)$, where $\mathcal{T}_{11}^{(i)} : \mathbb{C}^p \to \mathbb{C}$ are some functionals. We are going to prove that this pole is simple; clearly, it will complete the proof of the lemma.

Consider λ close to $\lambda(X)$ and not coinciding with λ_* and $\lambda(X)$. Let $f_i \in L_2(\mathbb{R}^n;\Omega_i)$ be arbitrary functions, $\boldsymbol{f} := (f_1,\dots,f_m) \in \boldsymbol{L}$, $\widetilde{f} := \sum_{i=1}^m \mathcal{S}(-X_i)f_i$. Completely by analogy with (4.1)–(4.6) one can check easily that the equation (3.5) with $\mathcal{T}_2 = \mathcal{T}_2^{(X)}$ is equivalent to

$$
\boldsymbol{g} + \mathcal{T}_7(\lambda)\boldsymbol{g} + \mathcal{T}_8(\lambda, X)\boldsymbol{g} = \boldsymbol{f}.
$$

Proceeding as in (4.9), (4.10), one can reduce this equation to an equivalent one

$$
\boldsymbol{g} - \sum_{i=1}^{p} \frac{\mathcal{T}_9^{(i)}\mathcal{T}_8(\lambda, X)\boldsymbol{g}}{\lambda - \lambda_*}\boldsymbol{\Phi}_i = -\sum_{i=1}^{p}\frac{\mathcal{T}_9^{(i)}\boldsymbol{f}}{\lambda - \lambda_*}\boldsymbol{\Phi}_i + \big(\mathrm{I} + \mathcal{T}_{10}(\lambda)\mathcal{T}_8(\lambda, X)\big)^{-1}\mathcal{T}_{10}(\lambda)\boldsymbol{f}. \quad (5.2)
$$

We denote

$$\kappa_i := \frac{\mathcal{T}_9^{(i)}\mathcal{T}_8(\lambda, X)\boldsymbol{g}}{\lambda - \lambda_*}$$

and apply the functionals $\mathcal{T}_9^{(j)}\mathcal{T}_8(\lambda, X)$ to (5.2). This procedure leads us to the equation for κ_i:

$$\mathrm{B}(\lambda, X)\boldsymbol{\kappa} = -\frac{1}{\lambda - \lambda_*}\mathrm{A}(\lambda, X)\boldsymbol{h}_1 + \boldsymbol{h}_2,$$

$$\boldsymbol{h}_1 := \begin{pmatrix} \mathcal{T}_9^{(1)}\boldsymbol{f} \\ \vdots \\ \mathcal{T}_9^{(p)}\boldsymbol{f} \end{pmatrix}, \quad \boldsymbol{h}_2 := \begin{pmatrix} \mathcal{T}_9^{(1)}\mathcal{T}_8(\lambda, X)\big(\mathrm{I} + \mathcal{T}_{10}(\lambda)\mathcal{T}_8(\lambda, X)\big)^{-1}\mathcal{T}_{10}(\lambda)\boldsymbol{f} \\ \vdots \\ \mathcal{T}_9^{(p)}\mathcal{T}_8(\lambda, X)\big(\mathrm{I} + \mathcal{T}_{10}(\lambda)\mathcal{T}_8(\lambda, X)\big)^{-1}\mathcal{T}_{10}(\lambda)\boldsymbol{f} \end{pmatrix}, \tag{5.3}$$

where $\boldsymbol{\kappa}$ is defined as in (4.15). Hence,

$$\boldsymbol{\kappa} = \frac{1}{\lambda - \lambda_*}\boldsymbol{h}_1 + \widetilde{\boldsymbol{\kappa}}, \quad \widetilde{\boldsymbol{\kappa}} := \mathrm{B}^{-1}\widetilde{\boldsymbol{h}}, \quad \widetilde{\boldsymbol{h}} := \boldsymbol{h}_2 - \boldsymbol{h}_1,$$

$$\boldsymbol{g} = \sum_{i=1}^{p} \widetilde{\kappa}_i \boldsymbol{\Phi}_i + \big(\mathrm{I} + \mathcal{T}_{10}(\lambda)\mathcal{T}_8(\lambda, X)\big)^{-1}\mathcal{T}_{10}(\lambda)\boldsymbol{f},$$

where $\widetilde{\kappa}_i$ are components of the vector $\widetilde{\boldsymbol{\kappa}}$. In accordance with Lemma 3.2 the solution to the equation (3.5) with $\mathcal{T}_2 = \mathcal{T}_2^{(X)}$ has at most simple pole at $\lambda(X)$. Hence, the same is true for the vector $\boldsymbol{g}$ just determined. It follows that the vector $\mathrm{B}^{-1}\widetilde{\boldsymbol{h}}$ can have at most simple pole at $\lambda(X)$. The estimates (4.7) imply that

$$\widetilde{\boldsymbol{h}} = -\boldsymbol{h}_1 + \mathcal{O}\big(l_X^{-\frac{n-1}{2}}\mathrm{e}^{-l_X\sqrt{-\lambda}}\big).$$

In view of this identity and the definition of $\boldsymbol{h}_1$ we conclude that for any $\widetilde{\boldsymbol{h}} \in \mathbb{C}^p$ there exists $\boldsymbol{f} \in \boldsymbol{L}$ such that $\widetilde{\boldsymbol{h}} = \boldsymbol{h}_2 - \boldsymbol{h}_1$, where $\boldsymbol{h}_i$ are given by (5.3). Therefore, the matrix B^{-1} has the simple pole at $\lambda(X)$. $\square$

Reproducing word for word the proof of Lemma 5.3 in [1] we obtain

Lemma 5.4. *A zero $\lambda(X) \xrightarrow[l_X \to +\infty]{} \lambda_*$ of the function $F(\lambda, X)$ has order q if and only if it is a q-multiple eigenvalue of $\mathcal{H}_X$.*

The statement of Theorem 1.2 follows from Lemmas 4.3, 4.4, 5.1, 5.4.

The proof of Theorems 1.3, 1.4 repeats *verbatim et literatim* the proof of Theorems 1.4, 1.5 in [1].

6 Proof of Theorems 1.5, 1.7 and Corollary 1.6

Proof of Theorem 1.5. Let us prove first that the representation (1.6) is valid, where the matrix A_0 is defined in the statement of the theorem and

$$\|\mathrm{A}_1\| = \mathcal{O}\left(l_X^{-n+1}\mathrm{e}^{-2l_X\sqrt{-\lambda_*}}\right), \quad l_X \to +\infty.$$

Due to (4.7), (4.13) we have

$$A_{i,j}(\lambda_*,X)=\mathcal{T}_9^{(i)}\mathcal{T}_8(\lambda_*,X)\boldsymbol{\phi}_j+\mathcal{O}\left(l_X^{-n+1}\mathrm{e}^{-2l_X\sqrt{-\lambda_*}}\right),\quad l_X\to+\infty.$$

We are going to show that $A_{i,j}^{(0)}(X)=\mathcal{T}_9^{(i)}\mathcal{T}_8(\lambda_*,X)\boldsymbol{\phi}_j$ and the matrix A_0 satisfies the condition (A); this will obviously imply the needed representation.

We choose some i and j and let $k,r\in\{1,\ldots,m\}$, $q\in\{1,\ldots,p_k\}$, $s\in\{1,\ldots,p_r\}$ be such that $i=\alpha_k+q$, $j=\alpha_r+s$. Then

$$\mathcal{T}_9^{(i)}\mathcal{T}_8(\lambda_*,X)\boldsymbol{\phi}_j=\begin{cases}\big(\mathcal{T}_6^{(k,r)}(\lambda_*)\phi_{r,s},\psi_{k,q}\big)_{L_2(\Omega_k)}, & r\neq k,\\ 0, & r=k.\end{cases}$$

Consider the case $r\neq k$. We employ (4.4) and (1.1) and integrate by parts to obtain:

$$\begin{aligned}\big(\mathcal{T}_6^{(k,r)}(\lambda_*)\phi_{r,s},\psi_{k,q}\big)_{L_2(\Omega_k)}&=\big(\mathcal{L}_k\mathcal{T}_5(\lambda_*,X_{k,r})\phi_{r,s},\psi_{k,q}\big)_{L_2(\Omega_k)}\\&+\big((\Delta-\lambda_*+\mathcal{L}_k)\chi_{\Omega_k}(\mathcal{H}_{\Omega_k}-\mathrm{i})^{-1}\mathcal{L}_k\mathcal{T}_5(\lambda,X_{k,r})\phi_{r,s},\psi_{k,q}\big)_{L_2(\Omega_k)}\\&=\big(\mathcal{L}_k\mathcal{T}_5(\lambda_*,X_{k,r})\phi_{r,s},\psi_{k,q}\big)_{L_2(\Omega_k)}.\end{aligned}\tag{6.1}$$

It follows from Lemma 3.1 and the definition of $\mathcal{T}_5$ that $\mathcal{T}_5(\lambda_*,X_{k,r})\phi_{k,s}=\mathcal{S}(X_{k,r})\psi_{r,s}$. Hence,

$$\mathcal{T}_9^{(i)}\mathcal{T}_8(\lambda_*,X)\boldsymbol{\phi}_j=\big(\mathcal{L}_k\mathcal{S}(X_{k,r})\psi_{r,s},\psi_{k,q}\big)_{L_2(\Omega_k)}=A_{i,j}^{(0)}(X).$$

Using this identity, the condition (1.1) and the equation for $\psi_{r,s}$ and $\psi_{k,q}$, we check that

$$\begin{aligned}A_{i,j}^{(0)}(X)&=\big(\mathcal{L}_k\mathcal{S}(X_{k,r})\psi_{r,s},\psi_{k,q}\big)_{L_2(\Omega_k)}=\big(\mathcal{S}(X_{k,r})\psi_{r,s},\mathcal{L}_k\psi_{k,q}\big)_{L_2(\Omega_k)}\\&=\big(\psi_{r,s},\mathcal{S}(X_{r,k})\mathcal{L}_k\psi_{k,q}\big)_{L_2(\mathbb{R}^n)}=\big(\psi_{r,s},(\Delta+\lambda_*)\mathcal{S}(X_{r,k})\psi_{k,q}\big)_{L_2(\mathbb{R}^n)}\\&=\big((\Delta+\lambda_*)\psi_{r,s},\mathcal{S}(X_{r,k})\psi_{k,q}\big)_{L_2(\mathbb{R}^n)}=\big(\mathcal{L}_r\psi_{r,s},\mathcal{S}(X_{r,k})\psi_{k,q}\big)_{L_2(\Omega_r)}\\&=\overline{\big(\mathcal{L}_r\mathcal{S}(X_{r,k})\psi_{k,q},\psi_{r,s}\big)}_{L_2(\Omega_r)}=\overline{A}_{j,i}^{(0)}(X).\end{aligned}\tag{6.2}$$

Hence, the matrix A_0 is hermitian. The eigenvectors of A_0 are orthonormal in $\mathbb{C}^p$, and the determinant of the matrix formed by these vectors thus equals one. Therefore, the matrix A_0 satisfies the condition (A). Now it is sufficient to apply Theorem 1.4 to complete the proof. ☐

Proof of Corollary 1.6. In the case considered the matrix A_0 reads as follows:

$$\mathrm{A}_0=\begin{pmatrix}0 & \big(\mathcal{L}_1\mathcal{S}(X_{1,2})\psi_2,\psi_1\big)_{L_2(\Omega_1)}\\ \overline{\big(\mathcal{L}_1\mathcal{S}(X_{1,2})\psi_2,\psi_1\big)}_{L_2(\Omega_1)} & 0\end{pmatrix},$$

where we have taken in account the hermiticity of this matrix. The eigenvalues of A_0 are $\tau_1^{(0)}=-\left|\big(\mathcal{L}_1\mathcal{S}(X_{1,2})\psi_2,\psi_1\big)_{L_2(\Omega_1)}\right|$, $\tau_2^{(0)}=\left|\big(\mathcal{L}_1\mathcal{S}(X_{1,2})\psi_2,\psi_1\big)_{L_2(\Omega_1)}\right|$. Applying now Theorem 1.5, we complete the proof. ☐

Proof of Theorem 1.7. Theorem 1.3 implies that the eigenvalue $\lambda(X)$ has the asymptotic expansion (1.4), where $\tau(X) = A_{11}(\lambda_*, X)$. It follows from the definition of $\mathbf{\Phi}_1$ and the estimates (4.7) that

$$\mathbf{\Phi}_1(\cdot, \lambda_*, X) = \boldsymbol{\phi}_1 - \mathcal{T}_{10}(\lambda_*)\mathcal{T}_8(\lambda_*, X)\boldsymbol{\phi}_1 + \mathcal{O}\big(l_X^{-n+1}\mathrm{e}^{-2l_X\sqrt{-\lambda_*}}\big), \quad l_X \to +\infty.$$

Since $\mathcal{T}_9^{(1)}\mathcal{T}_8(\lambda_*, X)\boldsymbol{\phi}_1 = 0$, we infer that

$$A_{11}(\lambda_*, X) = -\mathcal{T}_9^{(1)}\mathcal{T}_8(\lambda_*, X)\mathcal{T}_{10}(\lambda_*)\mathcal{T}_8(\lambda_*, X)\boldsymbol{\phi}_1 + \mathcal{O}\big(l_X^{-\frac{3n-3}{2}}\mathrm{e}^{-3l_X\sqrt{-\lambda_*}}\big), \tag{6.3}$$

as $l_X \to +\infty$. By direct calculations we check that

$$\mathcal{T}_{10}(\lambda_*)\mathcal{T}_8(\lambda_*, X)\boldsymbol{\phi}_1 = \Big(0, \big(\mathrm{I} + \mathcal{T}_7^{(2)}(\lambda_*)\big)^{-1}\mathcal{T}_6^{(2,1)}\phi_1, \ldots, \big(\mathrm{I} + \mathcal{T}_7^{(m)}(\lambda_*)\big)^{-1}\mathcal{T}_6^{(m,1)}\phi_1\Big),$$

where $\phi_1 := \big(\mathcal{T}_1^{(1)}(\lambda_*)\big)^{-1}\psi_1$. Using this relation and proceeding in the same way as in (6.1), we obtain

$$\begin{aligned} &\mathcal{T}_9^{(1)}\mathcal{T}_8(\lambda_*, X)\mathcal{T}_{10}(\lambda_*)\mathcal{T}_8(\lambda_*, X)\boldsymbol{\phi}_1 \\ &\quad = \sum_{j=2}^{m}\Big(\mathcal{L}_1\mathcal{S}(X_{1,j})\mathcal{T}_1^{(j)}(\lambda_*)\big(\mathrm{I} + \mathcal{T}_7^{(j)}(\lambda_*)\big)^{-1}\mathcal{T}_6^{(j,1)}\phi_1, \psi_1\Big)_{L_2(\Omega_1)}. \end{aligned} \tag{6.4}$$

In accordance with Lemma 3.1 the function $\mathcal{T}_1^{(j)}(\lambda_*)\big(\mathrm{I} + \mathcal{T}_7^{(j)}(\lambda_*)\big)^{-1}\mathcal{T}_6^{(j,1)}\phi_1$, $j = 2, \ldots, m$, is a solution to the equation (3.1) with $\mathcal{H}_{\mathcal{L}} = \mathcal{H}_j$, $\lambda = \lambda_*$, $f = \mathcal{T}_6^{(j,1)}\phi_1$. Since

$$\mathcal{T}_6^{(j,1)}\phi_1 = \mathcal{L}_j\mathcal{T}_5(\lambda_*, X_{j,1})\phi_1 - (\mathcal{H}_j - \lambda_*)\chi_{\Omega_j}(\mathcal{H}_{\Omega_j} - \mathrm{i})^{-1}\mathcal{L}_j\mathcal{T}_5(\lambda_*, X_{j,1})\phi_1,$$

due to (4.4), and $\mathcal{L}_j\mathcal{T}_5(\lambda_*, X_{j,1})\phi_1 = \mathcal{L}_j\mathcal{S}(X_{j,1})\psi_1$ by Lemma 3.1, we infer that

$$\begin{aligned} \mathcal{T}_1^{(j)}(\lambda_*)\big(\mathrm{I} + \mathcal{T}_7^{(j)}(\lambda_*)\big)^{-1}\mathcal{T}_6^{(j,1)}\phi_1 = &(\mathcal{H}_j - \lambda_*)^{-1}\mathcal{L}_j\mathcal{S}(X_{j,1})\psi_1 \\ &- \chi_{\Omega_j}(\mathcal{H}_{\Omega_j} - \mathrm{i})^{-1}\mathcal{L}_j\mathcal{S}(X_{j,1})\psi_1. \end{aligned}$$

The support of the second term in the right-hand side of the obtained identity lies inside Ω_j^β. Bearing this fact in mind, from (6.4) we deduce

$$\begin{aligned} &\mathcal{T}_9^{(1)}\mathcal{T}_8(\lambda_*, X)\mathcal{T}_{10}(\lambda_*)\mathcal{T}_8(\lambda_*, X)\boldsymbol{\phi}_1 \\ &\qquad\qquad = \sum_{j=2}^{m}\Big(\mathcal{L}_1\mathcal{S}(X_{1,j})(\mathcal{H}_j - \lambda_*)^{-1}\mathcal{L}_j\mathcal{S}(X_{j,1})\psi_1, \psi_1\Big)_{L_2(\Omega_1)}. \end{aligned}$$

We substitute this identity and (6.3) into (1.4) and take into account that by (4.7)

$$\mathcal{T}_9^{(1)}\mathcal{T}_8(\lambda_*, X)\mathcal{T}_{10}(\lambda_*)\mathcal{T}_8(\lambda_*, X)\boldsymbol{\phi}_1 = \mathcal{O}\big(l_X^{-n+1}\mathrm{e}^{-2l_X\sqrt{-\lambda_*}}\big), \quad l_X \to +\infty.$$

This leads us to the claimed asymptotics for $\lambda(X)$.

Since $p = 1$, the system (4.14) reduces to an equation $(\lambda - \lambda_* - A_{11}(\lambda, X))\kappa_1 = 0$, which has the non-trivial solution $\kappa_1 = 1$. This identity and Theorem 1.3 imply the asymptotics for $\psi(x, X)$. □

7 Examples

In this section we will give some possible examples of the operators $\mathcal{L}_i$. Throughout this section we suppose that $\Omega_i \subset \mathbb{R}^n$ are given bounded domains with infinitely differentiable boundary. We will often omit the index "i" in the notations corresponding to i-th operator $\mathcal{L}_i$ writing simply $\mathcal{L}$, Ω, $\mathcal{H}$, etc.

1. Potential. The simplest example of the operator $\mathcal{L}$ is the multiplication by the compactly supported real-valued potential. This is a classical example but it seems that in the multiple-well case $m \geqslant 3$ the asymptotics expansions for the eigenvalues were not known.

2. Second order differential operator. A more general example is a differential operator of the form

$$\mathcal{L} = \sum_{i,j=1}^{n} b_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + b_0, \tag{7.1}$$

where the coefficients b_{ij} are piecewise continuously differentiable and the coefficients b_i are piecewise continuous. The functions b_{ij} and b_i are also assumed to be complex-valued and compactly supported. We also suppose that the conditions (1.1), (1.2) hold true; the self-adjointness of the operator $\mathcal{H}$ and $\mathcal{H}_X$ follows from these conditions due to specific definition of $\mathcal{L}$.

The particular case of (7.1) is

$$\mathcal{L} = \operatorname{div} \mathrm{G}\nabla + \mathrm{i} \sum_{i=1}^{n} \left(b_i \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_i} b_i \right) + b_0, \tag{7.2}$$

where $\mathrm{G} = \mathrm{G}(x)$ is $n \times n$ hermitian matrix having piecewise continuously differentiable elements, the functions $b_i = b_i(x)$ are real-valued and piecewise continuously differentiable, the potential $b_0 = b_0(x)$ is a real-valued and piecewise continuous. We also suppose that the matrix G and the functions b_i are compactly supported and

$$|(\mathrm{G}(x)y, y)_{\mathbb{C}^n}| \leqslant c_0 \|y\|^2_{\mathbb{C}^n}, \qquad x \in \overline{\Pi}, \quad y \in \mathbb{C}^n,$$

where the constant c_0 is independent of x, y and obeys (1.3). The matrix G can be zero; in the case the operator $\mathcal{L}$ is a first order differential operator.

3. Magnetic Schrödinger operator. Let $\boldsymbol{b} = (b_1, \ldots, b_n) \in C_0^1(\mathbb{R}^n)$ be a magnetic vector-potential, and $b_0 := \|\boldsymbol{b}\|_{\mathbb{R}^n} + V$, where $V = V(x) \in C_0^\infty(\mathbb{R}^n)$ is an electric potential. We define the operator $\mathcal{L}$ by the formula (7.2) with $G = 0$. Such operator describes the magnetic field with compactly supported vector-potential.

4. Integral operator. The operator $\mathcal{L}$ is not necessary to be a differential one. For instance, it can be an integral operator of the form

$$(\mathcal{L}u)(x) := \int\limits_{\Omega} L(x, y) u(y) \, \mathrm{d}y,$$

where the kernel L is an element of $L_2(\Omega\times\Omega)$. We also assume that the function $L(\cdot,y)$ is compactly supported and the relation $L(x,y)=\overline{L(y,x)}$ holds true. Such operator satisfies the conditions (1.1), (1.2). It is also $\Delta_{\mathbb{R}^n}$-compact and therefore the operator $\mathcal{H}$ is self-adjoint.

5. δ-potential. The results of the general scheme developed in the present article can be applied to the perturbing operators not even satisfying the conditions we impose on $\mathcal{L}$. It is possible if such operators can be reduced by some transformations to an operator $\mathcal{L}$ satisfying needed restrictions. One of such examples is δ-potential supported by a manifold. Namely, let Γ be a bounded closed C^3-manifold in $\mathbb{R}^n$ of codimension one and oriented by a normal vector-field $\boldsymbol{\nu}=\boldsymbol{\nu}(\xi)$, where $\xi=(\xi_1,\ldots,\xi_n)$ are local coordinates on Γ. Let ϱ be the distance from a point to Γ measured in the direction of $\boldsymbol{\nu}$. We suppose that Γ is so that the coordinates (ϱ,ξ) are well-defined in a some neighbourhood of Γ, and in this neighbourhood the mapping $(\varrho,\xi)\mapsto x$ is C^3-diffeomorphism. We introduce the operator $\mathcal{H}_\Gamma:=-\Delta_{\mathbb{R}^n}+b\delta(x-\Gamma)$ as

$$\mathcal{H}_\Gamma v=-\Delta v,\quad x\notin\Gamma,$$

on the functions $v\in W_2^2(\mathbb{R}^n\setminus\Gamma)\cap W_2^1(\mathbb{R}^n)$ satisfying the condition

$$\frac{\partial v}{\partial\varrho}\bigg|_{\varrho=+0}-\frac{\partial v}{\partial\varrho}\bigg|_{\varrho=-0}=bv\big|_{\varrho=0},$$

where ϱ is $b=b(\xi)\in C^3(\Gamma)$. We reproduce now word for word the arguments of Example 5 in [1, Sec. 7] to establish

Lemma 7.1. *There exists C^1-diffeomorphism $\mathcal{P}:\mathbb{R}^n\to\mathbb{R}^n$, $\mathcal{P}=(\mathcal{P}_1,\ldots,\mathcal{P}_n)$, such that*

1. *The second derivatives of $\mathcal{P}$ and $\mathcal{P}^{-1}$ exist and are piecewise continuous.*
2. *The function $\mathrm{p}:=\det\mathrm{P}$ and the matrix*

$$\mathrm{P}:=\begin{pmatrix}\frac{\partial\mathcal{P}_1}{\partial x_1} & \cdots & \frac{\partial\mathcal{P}_1}{\partial x_n}\\ \vdots & & \vdots\\ \frac{\partial\mathcal{P}_n}{\partial x_1} & \cdots & \frac{\partial\mathcal{P}_n}{\partial x_n}\end{pmatrix}$$

satisfy the identities

$$\begin{gathered}\mathrm{p}^{1/2}\big|_{\varrho=+0}-\mathrm{p}^{1/2}\big|_{\varrho=-0}=0,\quad \frac{\partial}{\partial\varrho}\mathrm{p}^{1/2}\big|_{\varrho=+0}-\frac{\partial}{\partial\varrho}\mathrm{p}^{1/2}\big|_{\varrho=-0}=b,\\ \mathrm{P}\equiv\mathrm{E},\quad \mathrm{p}\equiv1\quad \text{as}\quad |\varrho|\geqslant\varepsilon,\end{gathered}\tag{7.3}$$

where ε is a some small fixed number.

3. The mapping $(\mathcal{U}v)(x) := \mathrm{p}^{-1/2}v(\mathcal{P}^{-1}(x))$ *is a linear unitary operator in* $L_2(\mathbb{R}^n)$ *which maps the domain of the operator* $\mathcal{H}_\Gamma$ *onto* $W_2^2(\mathbb{R}^n)$*. The identity*

$$\mathcal{H}_{\mathcal{L}} := \mathcal{U}\mathcal{H}_\Gamma\mathcal{U}^{-1} = -\Delta_{R^n} + \mathcal{L} \tag{7.4}$$

holds true, where the operator $\mathcal{L}$ *is given by (7.1) and the supports of* $b_{i,j}$*,* b_i *lie inside* $\{x : \rho \leqslant \varepsilon\}$*.*

The item 3 of this lemma implies that the original δ-potential can be reduced to a differential operator (7.1) without changing the spectrum. Thus, after such transformation we can apply the results of this paper to such perturbation as well.

The operator $\mathcal{L}$ in (7.4) depends on the auxiliary transformation $\mathcal{P}$. We are going to show that the leading terms of the asymptotics expansions established in Theorems 1.5, 1.7 and Corollary 1.6 do not depend of $\mathcal{P}$.

We begin with Theorem 1.5. Let $\mathcal{L}_k = \mathcal{L}$ for some k, where $\mathcal{L}$ is from (7.4), and $\widetilde{\psi}$ be an eigenfunction of the operator $\mathcal{H}_{\mathcal{L}}$ associated with λ_*. The corresponding elements of the matrix A_0 introduced in Theorem 1.5 are

$$A_{i,j}^{(0)} = (\mathcal{L}u, \widetilde{\psi})_{L_2(\Omega_{2\varepsilon})},$$

where $u = S(X_{k,r})\psi_{r,s}$, and $\Omega_{2\varepsilon} := \{x : \varrho < 2\varepsilon\}$. The function u satisfies the equation

$$(\Delta + \lambda_*)u = 0, \quad x \in \Omega_{2\varepsilon}. \tag{7.5}$$

The function $\psi := \mathcal{U}^{-1}\widetilde{\psi} = p^{1/2}\widetilde{\psi}(\mathcal{P}(\cdot))$ is an eigenfunction of $\mathcal{H}_\Gamma$ associated with λ_* and therefore it is independent on $\mathcal{P}$. The identities (7.3) imply that $\widetilde{\psi} \equiv \psi$ as $\varepsilon < \varrho \leqslant 2\varepsilon$. Employing this fact, (1.1), (7.5) and integrating by parts, we obtain

$$(\mathcal{L}u, \widetilde{\psi})_{L_2(\Omega_{2\varepsilon})} = (u, \mathcal{L}\widetilde{\psi})_{L_2(\Omega_{2\varepsilon})} = (u, (\Delta + \lambda_*)\widetilde{\psi})_{L_2(\Omega_{2\varepsilon})} = \int\limits_{\partial\Omega_{2\varepsilon}} \left(u\frac{\partial\overline{\psi}}{\partial\boldsymbol{\nu}_\varepsilon} - \overline{\psi}\frac{\partial u}{\partial\boldsymbol{\nu}_\varepsilon} \right) \mathrm{d}s,$$

where $\boldsymbol{\nu}_\varepsilon$ is the outward normal to $\partial\Omega_{2\varepsilon}$. The last integral is independent on ε since for any $\widetilde{\varepsilon} \in (0, \varepsilon)$

$$(\Delta + \lambda_*)\psi = 0, \quad x \in \Omega_{2\varepsilon} \setminus \Omega_{2\widetilde{\varepsilon}},$$

$$0 = (u, (\Delta + \lambda_*)\widetilde{\psi})_{L_2(\Omega_{2\varepsilon})\setminus\Omega_{2\widetilde{\varepsilon}}} = \int\limits_{\partial\Omega_{2\varepsilon}} \left(u\frac{\partial\overline{\psi}}{\partial\boldsymbol{\nu}_\varepsilon} - \overline{\psi}\frac{\partial u}{\partial\boldsymbol{\nu}_\varepsilon} \right) \mathrm{d}s - \int\limits_{\partial\Omega_{2\widetilde{\varepsilon}}} \left(u\frac{\partial\overline{\psi}}{\partial\boldsymbol{\nu}_{\widetilde{\varepsilon}}} - \overline{\psi}\frac{\partial u}{\partial\boldsymbol{\nu}_{\widetilde{\varepsilon}}} \right) \mathrm{d}s.$$

Using now the boundary conditions for ψ on Γ, we pass to the limit $\varepsilon \to +0$ and obtain

$$\lim_{\varepsilon\to 0} \int\limits_{\partial\Omega_{2\varepsilon}} \left(u\frac{\partial\overline{\psi}}{\partial\boldsymbol{\nu}_\varepsilon} - \overline{\psi}\frac{\partial u}{\partial\boldsymbol{\nu}_\varepsilon} \right) \mathrm{d}s = \int\limits_\Gamma u \left(\frac{\partial\overline{\psi}}{\partial\varrho}\bigg|_{\varrho=+0} - \frac{\partial\overline{\psi}}{\partial\varrho}\bigg|_{\varrho=-0} \right) \mathrm{d}s = (u, b\psi)_{L_2(\Gamma)}.$$

Thus, if an operator $\mathcal{L}_k$ describes the δ-potential, the corresponding elements of the matrix A_0 in Theorem 1.5 are

$$A_{i,j}^{(0)} := \big(\mathcal{S}(X_{k,r})\psi_{r,s}, \psi_{k,q}\big)_{L_2(\Gamma)},$$

where $\psi_{k,q}$ are the eigenfunctions of the operator $\mathcal{H}_\Gamma$. In particular, if in Corollary 1.6 the operator $\mathcal{H}_1$ is $\mathcal{H}_\Gamma$, the asymptotics expansions for λ_i become

$$\begin{aligned}
\lambda_1 &= \lambda_* - \left|\big(\mathcal{L}_1\mathcal{S}(X_{1,2})\psi_2, \psi_1\big)_{L_2(\Gamma)}\right| + \mathcal{O}\left(l_X^{-n+2}\mathrm{e}^{-2l_X\sqrt{-\lambda_*}}\right), \quad l_X \to +\infty,\\
\lambda_2 &= \lambda_* + \left|\big(\mathcal{L}_1\mathcal{S}(X_{1,2})\psi_2, \psi_1\big)_{L_2(\Gamma)}\right| + \mathcal{O}\left(l_X^{-n+2}\mathrm{e}^{-2l_X\sqrt{-\lambda_*}}\right), \quad l_X \to +\infty.
\end{aligned}$$

If under the hypothesis of Theorem 1.7 the operator $\mathcal{L}_1$ describes δ-potential, the arguments same as given above show that the asymptotics for $\lambda(X)$ reads as follows

$$\lambda(X) = \lambda_* - \sum_{j=2}^{m} \big(\mathcal{S}(X_{1,j})(\mathcal{H}_j - \lambda_*)^{-1}\mathcal{L}_j\mathcal{S}(X_{j,1})\psi_1, \psi_1\big)_{L_2(\Gamma)} + \mathcal{O}\big(l_X^{-\frac{3n-5}{2}}\mathrm{e}^{-3l_X\sqrt{-\lambda_*}}\big),$$

where ψ_1 is the eigenfunction of $\mathcal{H}_\Gamma$. The asymptotics for the associated eigenfunction remains the same if by ψ_1 we mean the eigenfunction of $\mathcal{H}_\Gamma$.

Suppose now that under the hypothesis of Theorem 1.7 one of the operators $\mathcal{L}_j$, $j \geqslant 2$, describes the δ-potential. We denote $u := (\mathcal{H}_j - \lambda_*)^{-1}\mathcal{L}_j\mathcal{S}(X_{j,1})\psi_1$. Proceeding in the same way as in (6.2), we obtain

$$\begin{aligned}
\big(\mathcal{L}_1\mathcal{S}(X_{1,j})u, \psi_1\big)_{L_2(\Omega_1)} &= \big((\Delta + \lambda_*)u, \mathcal{S}(X_{j,1})\psi_1\big)_{L_2(\Omega_{2\varepsilon})}\\
&= \int\limits_{\partial\Omega_{2\varepsilon}} \left(\mathcal{S}(X_{j,1})\overline{\psi}_1\frac{\partial u}{\partial\boldsymbol{\nu}_\varepsilon} - u\frac{\partial}{\partial\boldsymbol{\nu}_\varepsilon}\big(\mathcal{S}(X_{j,1})\overline{\psi}_1\big)\right)\mathrm{d}s.
\end{aligned} \tag{7.6}$$

Since $(\Delta + \lambda_*)\mathcal{S}(X_{j,1})\psi_1 = 0$ in $\Omega_{2\varepsilon}$, it follows that

$$\begin{aligned}
\mathcal{L}_j\mathcal{S}(X_{j,1})\psi_1 &= (\mathcal{H}_j - \lambda_*)\mathcal{S}(X_{j,1})\psi_1 + (\Delta + \lambda_*)\mathcal{S}(X_{j,1})\psi_1\\
&= \mathcal{U}\left((\mathcal{H}_\Gamma - \lambda_*)\mathcal{U}^{-1} + (\Delta + \lambda_*)\right)\mathcal{S}(X_{j,1})\psi_1.
\end{aligned}$$

Using this relation, (7.3), and the identity $(\mathcal{H}_j - \lambda_*)^{-1} = \mathcal{U}(H_\Gamma - \lambda_*)^{-1}\mathcal{U}^{-1}$, we obtain $u = \mathcal{U}U$, where

$$\begin{aligned}
&U = (\mathcal{H}_\Gamma - \lambda_*)^{-1}\left((\mathcal{H}_\Gamma - \lambda_*)\mathcal{U}^{-1} + (\Delta + \lambda_*)\right)\mathcal{S}(X_{j,1})\psi_1 = \widetilde{U} + U_j,\\
&\widetilde{U} = \mathrm{p}^{1/2}\psi_1(\mathcal{P}(\cdot + X_{j,1})) - \mathcal{S}(X_{j,1})\psi_1,
\end{aligned}$$

and $U_j \in W_2^2(\mathbb{R}^n \setminus \Gamma) \cap W_2^1(\mathbb{R}^n)$ is the unique solution to the problem

$$\begin{gathered}
(\Delta + \lambda_*)U_j = 0, \quad x \in \mathbb{R}^n \setminus \Gamma,\\
\left.\frac{\partial U_j}{\partial\varrho}\right|_{\varrho=+0} - \left.\frac{\partial U_j}{\partial\varrho}\right|_{\varrho=-0} = bU_j\big|_{\varrho=0} - b\mathcal{S}(X_{j,1})\psi_1\big|_{\varrho=0}.
\end{gathered}$$

It follows from (7.3) that $\widetilde{U}=0$, $u=U_j$ as $\varepsilon\leqslant\varrho\leqslant 2\varepsilon$. Bearing these relations in mind, we substitute the obtained representation for u into (7.6) and continue our calculations:

$$\big(\mathcal{L}_1\mathcal{S}(X_{1,j})u,\psi_1\big)_{L_2(\Omega_1)}=\int\limits_{\partial\Omega_{2\varepsilon}}\left(\mathcal{S}(X_{j,1})\overline{\psi}_1\frac{\partial U_j}{\partial\boldsymbol{\nu}_\varepsilon}-U_j\frac{\partial}{\partial\boldsymbol{\nu}_\varepsilon}\big(\mathcal{S}(X_{j,1})\overline{\psi}_1\big)\right)\,\mathrm{d}s.$$

The right hand side of this identity is independent on small ε that allows us to pass to the limit $\varepsilon\to+0$ and obtain

$$\begin{aligned}\big(\mathcal{L}_1\mathcal{S}(X_{1,j})u,\psi_1\big)_{L_2(\Omega_1)}&=\int\limits_{\Gamma}\mathcal{S}(X_{j,1})\overline{\psi}_1\left(\frac{\partial U_j}{\partial\varrho}\bigg|_{\varrho=+0}-\frac{\partial U_j}{\partial\varrho}\bigg|_{\varrho=-0}\right)\,\mathrm{d}s\\&=\big(bU_j-b\mathcal{S}(X_{j,1})\psi_1,\mathcal{S}(X_{j,1})\psi_1\big)_{L_2(\Gamma)}.\end{aligned}$$

Finally, it leads us to the following formula

$$\begin{aligned}\lambda(X)=&\lambda_*-\big(bU_j-b\mathcal{S}(X_{j,1})\psi_1,\psi_1\big)_{L_2(\Gamma)}\\&-\sum_{\substack{k=2\\k\neq j}}^{m}\big(\mathcal{L}_1\mathcal{S}(X_{1,k})(\mathcal{H}_k-\lambda*)^{-1}\mathcal{L}_k\mathcal{S}(X_{k,1})\psi_1,\psi_1\big)_{L_2(\Omega_1)}+\mathcal{O}\big(l_X^{-\frac{3n-5}{2}}\mathrm{e}^{-3l_X\sqrt{-\lambda_*}}\big),\end{aligned}$$

being valid as $l_X\to+\infty$ if the operator $\mathcal{H}_j$ describes the δ-potential.

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